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BOOLEAN SHEPARD INTERPOLATION

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Abstract. Using Shepard univariate interpolation projectors which form the chains and boolean methods we construct Biermann-Shepard projector. We study the approximation order of Biermann-Shepard operator for two particular cases. The convergence of this operator is mark out by graphs and numerical examples.

1. **Preliminaries**

Let X, Y be the linear spaces on $\mathbb{R}$ or $\mathbb{C}$.

The linear operator P defined on space X is called projector if $P^2 = P$.

The operator $P^C = I - P$, where I is identity operator, is called the remainder projector of P.

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$. If P, Q are commutative projectors then we have

$$\mathcal{P}(P \oplus Q) = \mathcal{P}(P) \cup \mathcal{P}(Q) \tag{1}$$

If P_1, P_2 are projectors on space X, we define relation "$\leq$":

$$P_1 \leq P_2 \Leftrightarrow P_1 P_2 = P_1 \tag{2}$$

Let be $f \in \mathcal{C}(X \times Y)$ and $x \in X$. We define $f^x \in C(Y)$ by

$$f^x(t) = f(x,t) \ , \ t \in Y$$

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For $y \in Y$ we define ${}^{y}f \in C(X)$ by

$$ {}^{y}f(s) = f(s,y)\ ,\ s \in X $$

Let P be a linear and bounded operator on $C(X)$. The parametric extension P' of P is defined by

$$ (P'f)(x,y) = (P\,{}^{y}f)(x) \tag{3} $$

If Q is a linear and bounded operator on $C(Y)$, then the parametric extension Q'' of Q is defined by

$$ (Q''f)(x,y) = (Qf^{x})(y) \tag{4} $$

Proposition 1. *Let $r \in \mathbb{N}$, P_1,..., P_r univariate interpolation projectors on $C(X)$ and Q_1,..., Q_r univariate interpolation projectors on $C(Y)$. Let P'_1,..., P'_r, Q''_1,..., Q''_r be the corresponding parametric extension. We assume that*

$$ P_1 \leq P_2 \leq \cdots \leq P_r,\ Q_1 \leq Q_2 \leq \cdots \leq Q_r \tag{5} $$

Then

$$ B_r = P'_1Q''_r \oplus P'_2Q''_{r-1} \oplus \cdots \oplus P'_rQ''_1 \tag{6} $$

is projector and it has representation

$$ B_r = \sum_{m=1}^{r} P'_mQ''_{r+1-m} - \sum_{m=1}^{r-1} P'_mQ''_{r-m} \tag{7} $$

Moreover, we have

$$ B_r^C = P_r'^C + P_{r-1}'^C Q_1''^C + \cdots + P_1'^C Q_{r-1}''^C + Q_r''^C - (P_r'^C Q_1''^C + \cdots + P_1'^C Q_r''^C) \tag{8} $$

where $P^C = I - P$, I is identity operator.

For the proof of this proposition see [3].

Remark 2. *If P_1,..., P_r and Q_1,..., Q_r are Lagrange univariate operators which form the chains (i.e. satisfy the relation (5)) the operator B_r given by (6) is called Biermann interpolation projectors. In this article, we instead the Lagrange univariate operators by Shepard univariate operators.*

2. Main result

Let be the univariate interpolation projectors of Shepard type $P_1,\ldots,$ P_r, $Q_1,\ldots,$ Q_r which are given by relations

$$(P_m f)(x) = \sum_{i=1}^{k_m} A_{i,m}(x) f(x_i),\ 1 \le m \le r \tag{9}$$

$$(Q_n g)(y) = \sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y) g(y_j),\ 1 \le n \le r$$

The interpolation points satisfy

$$\{x_1,\ldots,\, x_{k_m}\} \subseteq [a,b] \text{ and } \{y_1,\ldots,\, y_{l_n}\} \subseteq [c,d]$$

with

$$1 \le k_1 < k_2 < \cdots < k_r \text{ and } 1 \le l_1 < l_2 < \cdots < l_r \tag{10}$$

The cardinal functions are given by

$$A_{i,m}(x) = \frac{|x-x_i|^{-\mu}}{\sum_{k=1,\ k\neq i}^{k_m} |x-x_k|^{-\mu}},\ 1 \le i \le k_m \tag{11}$$

$$\widetilde{A}_{j,n}(y) = \frac{|y-y_j|^{-\mu}}{\sum_{l=1,\ l\neq j}^{l_n} |y-y_l|^{-\mu}},\ 1 \le j \le l_n$$

with $\mu \in \mathbb{R}$ and satisfy the relations

$$A_{i,m}(x_\nu) = \delta_{i\nu},\ i,\nu = \overline{1,k_m}$$

$$\widetilde{A}_{j,n}(y_\sigma) = \delta_{j\sigma},\ j,\sigma = \overline{1,l_n}$$

and

$$\sum_{i=1}^{k_m} A_{i,m}(x) = 1$$

$$\sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y) = 1$$

Theorem 3. *The parametric extensions*

$$P_1',\ldots,P_r',\ Q_1'',\ldots,Q_r''$$

are bivariate interpolation projectors which form the chains

$$P'_1 \leq P'_2 \leq \cdots \leq P'_r,\ Q''_1 \leq Q''_2 \leq \cdots \leq Q''_r \tag{12}$$

Proof. Let be $1 \leq m_1 \leq m_2 \leq r$. From (10) we have

$$k_{m_1} \leq k_{m_2} \tag{13}$$

We have that

$$(P'_{m_1} P'_{m_2} f)(x,y) = \sum_{i_1=1}^{k_{m_1}} A_{i_1,m_1}(x) \sum_{i_2=1}^{k_{m_2}} A_{i_2,m_2}(x_{i_1}) f(x_{i_2}, y) \tag{14}$$

But

$$A_{i_2,m_2}(x_{i_1}) = \delta_{i_2,i_1} \tag{15}$$

From (13), (14) and (15) we have that

$$(P'_{m_1} P'_{m_2} f)(x,y) = \sum_{i_1=1}^{k_{m_1}} A_{i_1,m_1}(x) f(x_{i_1}, y) = (P'_{m_1} f)(x,y)$$

i.e. $P'_{m_1} \leq P'_{m_2}$. Thus the projectors $P'_1, \ldots, P'_r$ form the chain. Analogous $Q''_1, Q''_2, \ldots, Q''_r$ are projectors which form a chain. □

We have that

$$P'_m Q''_n = Q''_n P'_m,\ 1 \leq m, n \leq r$$

and the tensor product projector has the representation:

$$(P'_m Q''_n f)(x,y) = \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} A_{i,m}(x) \widetilde{A}_{j,n}(y) f(x_i, y_j)$$

with interpolation properties

$$(P'_m Q''_n f)(x_i, y_j) = f(x_i, y_j),\ 1 \leq i \leq k_m,\ 1 \leq j \leq l_n$$

The projectors $P'_1, \ldots, P'_r,\ Q''_1, \ldots, Q''_r$ generate a distributive lattice of projectors on $C([a,b] \times [c,d])$. A special element in this lattice is

$$B_r^S = P'_1 Q''_r \oplus P'_2 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_1,\ r \in \mathbb{N} \tag{16}$$

called Biermann-Shepard projector and which has the interpolation properties

$$(B_r^S f)(x_i, y_j) = f(x_i, y_j),\ 1 \le i \le k_m,\ 1 \le j \le l_{r+1-m},\ 1 \le m \le r$$

The set of interpolation points of the Biermann-Shepard projector given by (16) has the disjoint representation

$$\mathcal{P}(B_r^S) = \bigcup_{m=1}^{r} \bigcup_{n=0}^{r-m} \{(x_i, y_j) : k_{m-1} < i < k_m,\ l_{r-m-n} < j \le l_{r-m-n+1}\} \tag{17}$$

with $k_0 = 0$, $l_0 = 0$. The number of interpolation points of Biermann-Shepard operator B_r^S given by (16) is

$$\left|\mathcal{P}(B_r^S)\right| = \sum_{m=1}^{r} k_m (l_{r+1-m} - l_{r-m})$$

with $l_0 = 0$.

Using the disjoint representation (17) of interpolation set we obtain the Lagrange representation of Biermann-Shepard interpolant

$$B_r(f) = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=1+k_{m-1}}^{k_m} \sum_{j=1+l_{r-m-n}}^{l_{r+1-m-n}} f(x_i, y_j) S_{ij} \tag{18}$$

The cardinal functions of Biermann-Shepard interpolation projector are given by

$$S_{ij}(x,y) = \sum_{s=m}^{m+n} A_{i,s}(x) \widetilde{A}_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} A_{i,S}(x) \widetilde{A}_{j,r-s}(y), \tag{19}$$

with $k_{m-1} \le i \le k_m,\ l_{r-m-n} \le j \le l_{r+1-m-n},\ 0 \le n \le r-m,\ 1 \le m \le r.$

For the remainder term we can use formula (8) and integral representation of remainder [1]

$$(P_m^C f)(x) = \int_a^b \varphi_m(x,s) f'(s) \mathrm{d}s \tag{20}$$

$$(Q_n^C g)(y) = \int_c^d \psi_n(t,y) g'(t)\, \mathrm{d}t$$

where

$$\varphi_m(x,s) = (x-s)_+^0 - \sum_{i=1}^{k_m} A_{i,m}(x)(x_i - s)_+^0$$

$$\psi_n(y,t) = (y-t)_+^0 - \sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y)(y_j - t)_+^0$$

Also

$$\left|(P_m^C f)(x)\right| \leq H_m(x) M_1 f$$

$$\left|(Q_n^C g)(y)\right| \leq K_n(y) M_1 g$$

where

$$H_m(x) = x - \sum_{i=1}^{k_m} x_i A_i(x) + 2\sum_{i=1}^{k_m} A_i(x)(x_i - x)_+$$

$$K_n(y) = y - \sum_{j=1}^{l_n} y_j \widetilde{A_j}(y) + 2\sum_{j=1}^{l_n} \widetilde{A}_j(y)(y_j - y)_+$$

$$M_1 f = \sup_{a \leq x \leq b} |f'(x)|$$

$$M_1 g = \sup_{c \leq y \leq d} |g'(y)|$$

If $f \in C^{1,1}([a,b] \times [c,d])$ we have the following estimation for remainder term of Biermann-Shepard interpolant

$$\left|f(x,y) - B_r^S f(x,y)\right| \leq H_r(x) \left\|f^{(1,0)}\right\| + K_r(y) \left\|f^{(0,1)}\right\| + \sum_{i=1}^{r-1} H_{r-i}(x) K_i(y) \left\|f^{(1,1)}\right\|$$

$$+ \sum_{i=1}^{r} H_{r+1-i}(x) K_i(y) \left\|f^{(1,1)}\right\|$$

where $\left\|f^{(i,j)}\right\| = \max_{(x,y)\in[a,b]\times[c,d]} \left|f^{(i,j)}(x,y)\right|$.

3. **Examples**

Using relations (8) we determine the approximation order of Biermann-Shepard projector (16) for two particular case.

Example 1

Let be

$$k_m = 2^m + 1,\ 1 \le m \le r$$

$$l_n = 2^n + 1,\ 1 \le n \le r$$

and the univariate Shepard interpolation projectors on $[0,1]$ with equidistant nodes

$$(P_m f)(x) = (S_{2^m,\mu} f)(x) = \frac{\sum_{k=0}^{2^m} f(\frac{k}{2^m}) \left|x - \frac{k}{2^m}\right|^{-\mu}}{\sum_{k=0}^{2^m} \left|x - \frac{k}{2^m}\right|^{-\mu}},\ 1 \le \mathrm{m} \le \mathrm{r} \tag{21}$$

$$(Q_n g)(y) = (S_{2^n,\mu} g)(y) = \frac{\sum_{j=0}^{2^n} g(\frac{j}{2^n}) \left|y - \frac{j}{2^n}\right|^{-\mu}}{\sum_{j=0}^{2^n} \left|y - \frac{j}{2^n}\right|^{-\mu}}, 1 \le \mathrm{n} \le \mathrm{r}$$

We have that the extension projectors form the chains

$$P_1' \le P_2' \le \cdots \le P_r',\ Q_1'' \le Q_2'' \le \cdots \le Q_r''$$

and we can define the Biermann-Shepard operator

$$B_r^S = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1'' \tag{22}$$

From [5], if ${}^y f \in Lip_{[0,1]}1$, we have that

$$\|{}^y f - (S_{2^m,\mu}{}^y f)\| = \begin{cases} O(\frac{1}{2^m}) & \mu > 2 \\ O(\frac{m}{2^m}) & \mu = 2 \\ O(\frac{m}{2^{m(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{m}) & \mu = 1 \end{cases} \tag{23}$$

If $f^x \in Lip_{[0,1]}1$ we obtain a analogous estimation for $\|f^x - S_{2^n,\mu} f^x\|$.

Theorem 4. *If $f \in Lip_{[0,1]}1 \times Lip_{[0,1]}1$, the approximation orders of the B_r^S interpolant given by (22) are*

$$\|f - B_r^S f\| = \begin{cases} O(\frac{r}{2^r}) & , & \mu > 2 \\ O(\frac{r^3}{2^r}) & , & \mu = 2 \\ O(\frac{r^3}{2^{r(\mu-1)}}) & , & \mu \in (1,2) \\ O(\frac{1}{r}) & , & \mu = 1 \end{cases}$$

Proof. From (8) we have

$$(B_r^S)^C = (S'_{2^r,\mu})^C + (S''_{2^r,\mu})^C +$$

$$+ \sum_{m=1}^{r-1} (S'_{2^{r-m},\mu})^C (S''_{2^m,\mu})^C - \sum_{m=1}^{r} (S'_{2^{r+1-m},\mu})^C (S''_{2^m,\mu})^C$$

Taking into account (23) on obtain

- in the case $\mu > 2$

$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\le \frac{c}{2^r} + \frac{c}{2^r} + \sum_{m=1}^{r-1} \frac{c}{2^{r-m}} \cdot \frac{c}{2^m} + \sum_{m=1}^{r} \frac{c}{2^{r+1-m}} \cdot \frac{c}{2^m} = \\ &= O(\frac{r}{2^r}).\end{aligned}$$

- in the case $\mu = 2$

$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\le c\frac{r}{2^r} + c\frac{r}{2^r} + \sum_{m=1}^{r-1} \frac{c(r-m)}{2^{r-m}} \cdot \frac{cm}{2^m} + \sum_{m=1}^{r} \frac{c(r+1-m)}{2^{r+1-m}} \cdot \frac{cm}{2^m} = \\ &= O(\frac{r^3}{2^r}).\end{aligned}$$

- in the case $\mu \in (1,2)$

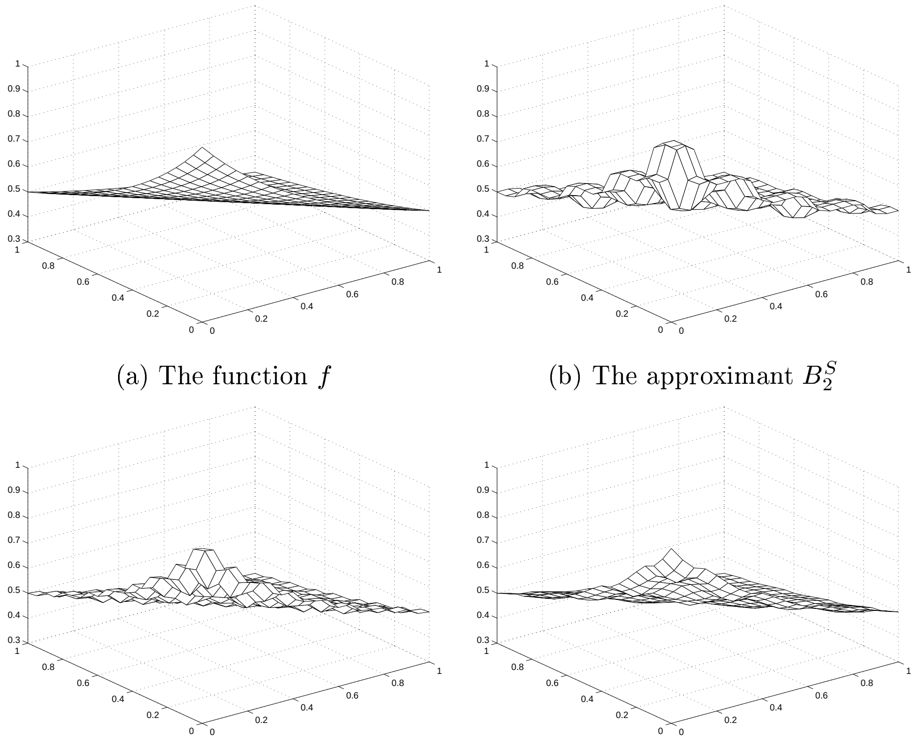
$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\le c\frac{r}{2^{r(\mu-1)}} + c\frac{r}{2^{r(\mu-1)}} + \sum_{m=1}^{r-1} \frac{c(r-m)}{2^{(r-m)(\mu-1)}} \cdot \frac{cm}{2^{m(\mu-1)}} + \\ &\quad + \sum_{m=1}^{r} \frac{c(r+1-m)}{2^{(r+1-m)(\mu-1)}} \cdot \frac{cm}{2^{m(\mu-1)}} \\ &= O(\frac{r^3}{2^{r(\mu-1)}}).\end{aligned}$$

- in the case $\mu = 1$

$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\le \frac{c}{r} + \frac{c}{r} + \sum_{m=1}^{r-1} \frac{c}{r-m} \cdot \frac{c}{m} + \sum_{m=1}^{r} \frac{c}{r+1-m} \cdot \frac{c}{m} = \\ &= O(\frac{1}{r}).\end{aligned}$$

□

In Figure 1 we approximate the functions $f : [0,1] \times [0,1] \to R$ $f(x,y) = \frac{1}{1+x+y}$ by B_r^S for $\mu = 4$ and $r = 2,3,4$.



(a) The function f

(b) The approximant B_2^S

(c) The approximant B_3^S

(d) The approximant B_4^S

FIGURE 1. The graph of function $f(x,y) = 1/(1+x+y)$ and the graphs of B_r^S for $\mu = 4$ and $r = 2, 3, 4$

We have the estimations

r	$\|f - B_r^S f\|$	$\|\mathcal{P}(B_r^S f)\|$
2	0.0998	21
3	0.0531	49
4	0.0295	113
5	0.0167	257

Remark 5. *Under stronger restrictions on f, see* [5], [6]

$$f^{(1,0)}(0,y) = f^{(1,0)}(1,y),\ y \in [0,1] \tag{24}$$
$$f^{(0,1)}(x,0) = f^{(0,1)}(x,1),\ x \in [0,1]$$
$$f \in C^{1,1}([0,1] \times [0,1])$$

we have

$$\|{}^y f - (S_{2^m,2}{}^y f)\| = O(\frac{1}{2^m})$$
$$\|f^x - S_{2^n,2} f^x)\| = O(\frac{1}{2^n})$$

which implies

$$\left\| f - B_r^S f \right\| = O(\frac{r}{2^r}),\ \textit{for}\ \mu = 2 \tag{25}$$

Remark 6. *The approximation orders of product operator $S'_{2^r,\mu} S''_{2^r,\mu}$ are*

$$\left\| f - (S'_{2^r,\mu} S''_{2^r,\mu} f) \right\| = \begin{cases} O(\frac{1}{2^r}) & \mu > 2 \\ O(\frac{r}{2^r}) & \mu = 2 \\ O(\frac{r}{2^{r(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{r}) & \mu = 1 \end{cases}$$

But,

$$\left| \mathcal{P}(B_r^S) \right| = 2^r(r+3) + 1$$
$$\left| \mathcal{P}(S'_{2^r,\mu} S''_{2^r,\mu}) \right| = (2^r + 1)^2$$

It follows the Biermann-Shepard operators B_r^S is more efficient that operator $S'_{2^r,\mu} S''_{2^r,\mu}$.

Example 2

Let be $r = 2$ and

$$k_1 = N + 1,\ k_2 = N^2 + 1$$
$$l_1 = N + 1,\ l_2 = N^2 + 1$$

and the univariate Shepard interpolation projectors on $[0,1]$ with equidistant nodes

$$(P_m f)(x) = (S_{N^m,\mu} f)(x) = \frac{\sum_{k=0}^{N^m} f(\frac{k}{N^m}) \left| x - \frac{k}{N^m} \right|^{-\mu}}{\sum_{k=0}^{N^m} \left| x - \frac{k}{N^m} \right|^{-\mu}}, \; m = 1,2 \tag{26}$$

$$(Q_n g)(y) = (S_{N^n,\mu} g)(y) = \frac{\sum_{j=0}^{N^n} g(\frac{j}{N^n}) \left| y - \frac{j}{N^n} \right|^{-\mu}}{\sum_{j=0}^{N^n} \left| y - \frac{j}{N^n} \right|^{-\mu}}, \; n = 1,2$$

The Biermann-Shepard interpolation projector is given by

$$B_2^S = P_1' Q_2'' \oplus P_2' Q_1'' \tag{27}$$

If ${}^y f \in Lip_{[0,1]} 1$, we have that (from [5])

$$\|{}^y f - (S_{N^m,\mu} {}^y f)\| = \begin{cases} O(\frac{1}{N^m}) & \mu > 2 \\ O(\frac{\log N}{N^m}) & \mu = 2 \\ O(\frac{1}{N^{m(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{\log N}) & \mu = 1 \end{cases} \quad m = 1.2 \tag{28}$$

If $f^x \in Lip_{[0,1]} 1$ we obtain a analogous estimation for $\|f^x - S_{N^m,\mu} f^x\|$.

Theorem 7. *If $f \in Lip_{[0,1]} 1 \times Lip_{[0,1]} 1$, the approximation orders of the B_2^S interpolant given by (27) are*

$$\left\| f - B_2^S f \right\| = \begin{cases} O(\frac{1}{N^2}) & \mu > 2 \\ O(\frac{\log^2 N}{N^2}) & \mu = 2 \\ O(\frac{1}{N^{2(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{\log N}) & \mu = 1 \end{cases}$$

Proof. From (8) we have

$$(B_2^S)^C = (S'_{N^2,\mu})^C + (S''_{N^2,\mu})^C + (S'_{N,\mu})^C (S''_{N,\mu})^C$$

$$-(S'_{N^2,\mu})^C (S''_{N,\mu})^C - (S'_{N,\mu})^C (S''_{N^2,\mu})^C$$

Taking into account (28) on obtain

- in the case $\mu > 2$

$$\left\| (B_2^S f)^C f \right\| \le \frac{c}{N^2} + \frac{c}{N^2} + \frac{c}{N} \cdot \frac{c}{N} + \frac{c}{N^2} \cdot \frac{c}{N} + \frac{c}{N} \cdot \frac{c}{N^2} = O(\frac{1}{N^2})$$

• in the case $\mu = 2$

$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq c\frac{\log N}{N^2} + c\frac{\log N}{N^2} + c\frac{\log N}{N} \cdot c\frac{\log N}{N} + \\ &+c\frac{\log N}{N^2} \cdot c\frac{\log N}{N} + c\frac{\log N}{N} \cdot c\frac{\log N}{N^2} \\ &= O(\frac{\log^2 N}{N^2})\end{aligned}$$

• in the case $\mu \in (1,2)$

$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq \frac{c}{N^{2(\mu-1)}} + \frac{c}{N^{2(\mu-1)}} + \frac{c}{N^{(\mu-1)}} \cdot \frac{c}{N^{(\mu-1)}} + \\ &+\frac{c}{N^{2(\mu-1)}} \cdot \frac{c}{N^{(\mu-1)}} + \frac{c}{N^{(\mu-1)}} \cdot \frac{c}{N^{2(\mu-1)}} \\ &= O(\frac{1}{N^{2(\mu-1)}})\end{aligned}$$

• in the case $\mu = 1$

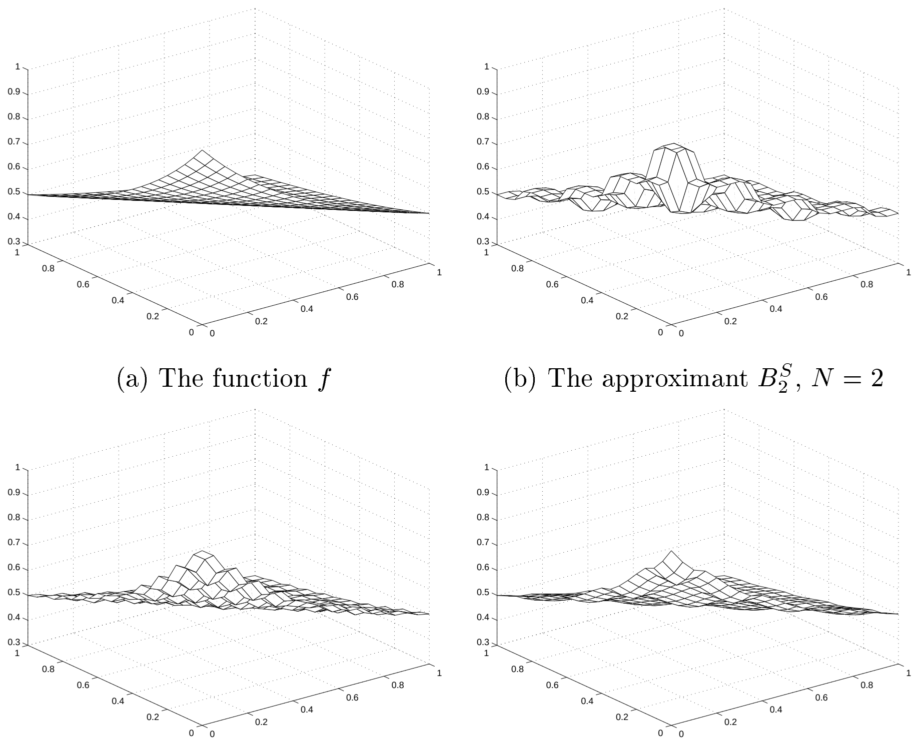
$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq \frac{c}{\log N} + \frac{c}{\log N} + \frac{c}{\log N} \cdot \frac{c}{\log N} + \\ &+\frac{c}{\log N} \cdot \frac{c}{\log N} + \frac{c}{\log N} \cdot \frac{c}{\log N} \\ &= O(\frac{c}{\log N})\end{aligned}$$

□

In Figure 2 we approximate the functions $f : [0,1] \times [0,1] \to R$ $f(x,y) = \frac{1}{1+x+y}$ by B_2^S for $\mu = 4$ and $N = 2, 3, 4$.

We have the following estimations

N	$\left\|f - B_2^S f\right\|$	$\left\|\mathcal{P}(B_2^S f)\right\|$
2	0.0998	21
3	0.0365	64
4	0.0283	145
5	0.0135	276



(a) The function f (b) The approximant B_2^S, $N = 2$

(c) The approximant B_2^S, $N = 3$ (d) The approximant B_2^S, $N = 4$

FIGURE 2. The graph of function $f(x, y) = 1/(1 + x + y)$ and the graphs of B_2^S for $\mu = 4$ and $N = 2, 3, 4$

Remark 8. *Under the same stronger restrictions on f given by (24) we have that*

$$\|^y f - (S_{N^m,2}{}^y f)\| = O(\frac{1}{N^m})$$
$$\|f^x - (S_{N^n,2} f^x)\| = O(\frac{1}{N^n})$$

which implies

$$\left\|f - B_2^S f\right\| = O(\frac{1}{N^2}),\ for\ \mu = 2. \quad (29)$$

Remark 9. *The approximation orders of operator* $S'_{N^2}S''_{N^2}$ *are*

$$\|f-(S'_{N^2}S''_{N^2}f)\|=\begin{cases} O(\frac{1}{N^2}) & \mu>2\\ O(\frac{\log N}{N^2}) & \mu=2\\ O(\frac{1}{N^{2(\mu-1)}}) & \mu\in(1,2)\\ O(\frac{1}{\log N}) & \mu=1\end{cases} \tag{30}$$

But,

$$\left|\mathcal{P}(S'_{N,\mu}S''_{N^2,\mu}\oplus S'_{N^2,\mu}S''_{N,\mu})\right| = 2N^3+N^2+1$$
$$\left|\mathcal{P}(S'_{N^2,\mu}S''_{N^2,\mu})\right| = (N^2+1)^2$$

It follows that the Biermann-Shepard operator B_r^S *given by (27) is more efficient that operator* $S'_{N^2}S''_{N^2}$.

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