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COST OF TRACKING FOR DIFFERENTIAL STOCHASTIC EQUATIONS IN HILBERT SPACES

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Abstract. We consider the tracking problem for differential stochastic equations diffusions dependent on both state and control variables. The Riccati equation associated with this problem is in general different from the conventional Riccati equation. We establish that under stabilizability and uniform observability conditions this equation has a unique positive and bounded on $\mathbf{R}_+$ solution. Using this result we find the optimal control (and the optimal cost) for tracking problem (see also [11]).

Notations and statement of the problem

Let H, U, V be separable real Hilbert spaces. Let $J \subset \mathbf{R}_+ = [0, \infty)$ be an interval. If E is a Banach space we denote by $C(J, E)$ the space of all mappings $G(t) : J \to E$ that are continuous. We also denote by $C_s(J, L(H))$ the space of all strongly continuous mappings $G(t) : J \to L(H)$ and by $C_b(J, L(H))$ the subspace of $C_s(J, L(H))$, which consist of all mappings $G(t)$ such that $\sup\limits_{t \in J} \|G(t)\| < \infty$. Given a signal $r \in C_b(\mathbf{R}_+, H)$ we want to minimize the cost

$$J(s,u) = \overline{\lim_{t\to\infty}} \frac{1}{t-s} E \int_s^t \|C(\sigma)\left(x(\sigma) - r(\sigma)\right)\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \, d\sigma$$

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in a suitable class of control u subject to the equation (denoted $\{A : B; G_i : H_i\}$)

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sum_{i=1}^{m} (G_i(t)x(t) + H_i(t)u(t))\, dw_i(t) \tag{1}$$

$$x(s) = x \in H.$$

We assume that the coefficients satisfy the following hypothesis:

$\mathbf{P}_1$: $A, G_i \in C_b(R_+, L(H)), i = 1, 2, ...m, m \in N^*$, $B, H_i \in C_b(R_+, L(U, H))$, $B^*, H_i^* \in C_b(R_+, L(H, U))$, $C \in C_b(R_+, L(H, V)), C^*C, G_i, G_i^* \in C_b(R_+, L(H))$, $K \in C_b(R_+, L(U))$ and there exist $\delta_0 > 0$ such that for all $t \in R_+, K(t) \geq \delta_0 I$. If $Z \in C_b(R_+, L(H, V))$, we will denote $\widetilde{Z} = \sup_{0 \leq r < \infty} \|Z(r)\| < \infty$.

1. Stabilizability, detectability and uniform observability

It is known (see Proposition 5 in [13] and Definition 5.3 in [4]) that if $A \in C_s(\mathbf{R}_+, L(H))$ then the family $A(t), t \geq 0$ generates an evolution operator $U(t, s)$ which has the following properties: 1. $(t, s) \to U(t, s)$ *is continuous in the uniform operator topology on* $\{(t, s)/0 \leq s \leq t \leq T\}$; *2.* $\frac{\partial U(t,s)x}{\partial t} = L(t)U(t, s)x$ *and* $\frac{\partial U(t,s)x}{\partial s} = -U(t, s)L(s)x$ *for all* $x \in H$ *and* $0 \leq s \leq t \leq T$.

In the sequel we will assume that P_1 holds if we don't specify other conditions. Let $(\Omega, F, F_t, t \in [0, \infty), P)$ be a stochastic basis. We consider the equation

$$dy(t) = A(t)y(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t),\ y(s) = x \in H, \tag{2}$$

denoted by $\{A; G_i\}$, where w_i's are independent real Wiener processes relative to F_t. It is known (see [5] and the notations therein) that (2) has a unique mild solution in $C([s, T]; L^2(\Omega; H))$ that is adapted to F_t; namely the solution of

$$y(t) = U(t, s)x + \sum_{i=1}^{m} \int_s^t U(t, r)G_i(r)y(r)dw_i(r). \tag{3}$$

This solution is also a strong solution, that is $y(t)$ satisfies the integral equation

$$y(t) = x + \int_s^t A(r)y(r)dr + \sum_{i=1}^{m} \int_s^t G_i(r)y(r)dw_i(r).$$

Definition 1. *Let $y(t,s;x)$ be the mild solution of $\{A;G_i\}$. We say that (2) is uniformly exponentially stable if there exist constants $M \geq 1, \omega > 0$ such that $E\left\|y(t,s;x)\right\|^2 \leq Me^{-\omega(t-s)}\left\|x\right\|^2$ for all $t \geq s \geq 0$ and $x \in H$.*

If $C \in C_s([0,\infty), L(H,V))$ we consider the system formed by the equation (2) and the observation relation $z(t) = C(t)y(t,s,x)$ denoted by $\{A, G_i; C\}$.

Definition 2. *(see* [12]*) We say that the system $\{A, C; G_i\}$ is uniformly observable if there exist $\tau > 0$ and $\gamma > 0$ such that $E\int\limits_s^{s+\tau}\left\|C(t)y(t,s;x)\right\|^2 dt \geq \gamma\left\|x\right\|^2$ for all $s \in R_+$ and $x \in H$.*

Definition 3. *(see* [5]*) We say that the system $\{A, C; G_i\}$ is detectable if there exists $L \in C_b([0,\infty), L(V,H))$ such that $\{A + LC; G_i\}$ is uniformly exponentially stable.*

Definition 4. *We say that $\{A : B; G_i : H_i\}$ is stabilizable if there exists $F \in C_b([0,\infty), L(H,U))$ such that $\{A + BF; G_i + H_iF\}$ is uniformly exponentially stable.*

In the deterministic case it is known (see [7] for the autonomous case) that uniform observability implies detectability. We proved in [12] that this assertion is not true in the stochastic case.

2. **Bounded solutions of Riccati equation of stochastic control**

Let us consider the linear and bounded operator
$\mathcal{B}$: $C_s(\mathbf{R}_+, L(H)) \to C_s(\mathbf{R}_+, L(H,U)), \mathcal{B}(P)(s) = B^*(s)P(s) + \sum\limits_{i=1}^{m} H_i^*(s)P(s)G_i(s)$
and the function $\mathcal{K} : C_s(\mathbf{R}_+, L(H)) \to C_s(\mathbf{R}_+, L(U))$, $\mathcal{K}(P)(s) = K(s) + \sum\limits_{i=1}^{m} H_i^*(s)$ $P(s)H_i(s)$. Since K is uniformly positive, then it is easy to see that $\mathcal{K}(P)$ is uniformly positive. We consider the following Riccati equation in $C_s([0,\infty), L^+(H))$

$$P' + A^*P + PA + \sum_{i=1}^{m} G_i^*PG_i + C^*C - \left[\mathcal{B}(P)\right]^*\left[\mathcal{K}(P)\right]^{-1}\mathcal{B}(P) = 0, \tag{4}$$

where the weak differentiability is considered. If $P \in C_s([0,\infty), L^+(H))$ we put

$$S(s) = -\left[\mathcal{K}(P)(s)\right]^{-1}\mathcal{B}(P)(s), s \geq 0. \tag{5}$$

and we denote $\widehat{A} = A - BS$, $\widehat{G}_i = G_i - H_i S$. Then (4) can be written as it follows

$$P' + \widehat{A}^* P + P\widehat{A} + \sum_{i=1}^{m} \widehat{G}_i^* P \widehat{G}_i + C^*C + S^*KS = 0. \tag{6}$$

Arguing as in the proof of Proposition 4.64 [3] and using Dini's theorem we can prove the following lemma.

Lemma 1. *If $(P_n)_{n\in N^*}$ is an increasing sequence in $C_s([0,T], L^+(H))$ such as $P_n(t) \leq I$, for all $t \in [0,T]$ (I is the identity operator on H), then there exists $P \in C_s([0,T], L^+(H))$ such as $P_n(t)x \underset{n\to\infty}{\to} P(t)x, x \in H$, uniformly for $t \in [0,T]$.*

Theorem 1. *The Riccati equation (4) with the final condition $P(T) = R \in L^+(H), T \in \mathbf{R}_+^*$ has a unique solution in $C_s([0,T], L^+(H))$ denoted $P(T,s;R)$, which also belongs to $C([0,T], L^+(H))$ and has the following properties:*

a) It is the unique solution of the integral equation

$$P(s)x = U^*(T,s)P(T)U(T,s)x + \int_s^T U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)P(r)G_i(r) \tag{7}$$

$$+C^*(r)C(r) - [\mathcal{B}(P)(r)]^* [\mathcal{K}(P)(r)]^{-1} \mathcal{B}(P)(r)]U(r,s)xdr.$$

b) It is monotone in the sense that $P(T,s;R_1) \leq P(T,s;R_2)$, if $R_1 \leq R_2$.

Proof. The existence of the solution. The proof is similar to that given in [1] for the finite dimensional case. We consider the following iterative scheme to construct the solution of (6). Let $P_0 = I$ (I is the identity operator on H), $S_0 = -[\mathcal{K}(P_0)]^{-1} \mathcal{B}(P_0), \widehat{A}_0 = A - BS_0$, $\widehat{G}_{0,i} = G_i - H_i S_0$, $i = 1,..,m$. Using Lemma 1 in [7] we deduce that the following differential equation

$$P'_{n+1} + \widehat{A}_n^* P_{n+1} + P_{n+1}\widehat{A}_n + \sum_{i=1}^{m} \widehat{G}_{n,i}^* P_{n+1} \widehat{G}_{n,i} + C^*C + S_n^* K S_n = 0, \tag{8}$$

$$P_{n+1}(T) = R,$$

where $S_n = -[\mathcal{K}(P_n)]^{-1} \mathcal{B}(P_n), \widehat{A}_n = A - BS_n$, $\widehat{G}_{n,i} = G_i - H_i S_n, i = 1,..,m$, $n = 0,1,2,...$ has a unique solution which belongs to $C_s([0,T], L^+(H))$. As in [1] we can establish that $\{P_n(.)\}$ is a decreasing sequence. Using the above lemma for the increasing sequence $\{I - P_n(.)\}$, it follows that there exists $P \in C_s([0,T], L^+(H))$

such that, for all $x \in H$, $P_n(t)x \underset{n\to\infty}{\to} P(t)x$, uniformly for $t \in [0,T]$. As $n \to \infty$ in (8) we deduce that P is weakly differentiable and satisfies (6). Thus (4) with the final condition $P(T) = R \in L^+(H)$, has a solution in $C_s(\mathbf{R}_+, L^+(H))$. Differentiating the function $f_x : [0,T] \to \mathbf{R}$ $f_x(\sigma) = \langle P(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle$ we get

$$\frac{\partial f_x(\sigma)}{\partial \sigma} = \langle P'(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle + 2\langle P(\sigma)A(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle .$$

Now, we integrate from s to T, $s \in [0,T]$ the above relation and we obtain (7). Using the Gronwall's lemma we deduce that (7) has a unique solution in $C_s(\mathbf{R}_+, L^+(H))$, and consequently (4) has a unique solution in $C_s(\mathbf{R}_+, L^+(H))$. It is not difficult to see that a solution of (7) belongs to $C(\mathbf{R}_+, L^+(H))$. Thus (4) has a unique solution in $C(\mathbf{R}_+, L^+(H))$ and a) holds.

Now we prove b).Let $R, R_1 \in L^+(H), R_1 \le R$ and let $P(s) = P(T,s;R)$, $P_1(s) = P(T,s;R_1)$ be the corresponding solutions of (4). We use the notations $\Delta = P - P_1$, $S_1 = -[\mathcal{K}(P_1)]^{-1}\mathcal{B}(P_1)$, $\widehat{A}_1 = A - BS_1$, $\widehat{G}_{1,i} = G_i - H_iS_1$. Then, Δ is the solution of the following Lyapunov equation with the final condition $\Delta(T) = R - R_1$

$$\Delta' + \widehat{A}_1^*\Delta + \Delta\widehat{A}_1 + \sum_{i=1}^{m}\widehat{G}_{1,i}^*\Delta\widehat{G}_{1,i} + (S_1 - S)^*\mathcal{K}(P)(S_1 - S) = 0. \tag{9}$$

Thus it follows that $\Delta \ge 0$ and $P - P_1 \ge 0$ and we obtain the conclusion. □

Remark 1. *The function $F : [0,T] \to R, F(t,x) = \langle P(t)x, x\rangle$, where $P(t) = P(T,t;R)$ and the strong solution of (2) satisfy the conditions required by Ito's formula in infinite dimensions (see T. 3.8 in* [2]*).*

Moreover, if $P \in C_s(\mathbf{R}_+, L^+(H))$ is a solution of (4) and $\sup_{s\in\mathbf{R}_+}\|P(s)\| < \infty$, then P is said to be a *bounded solution.* Assume that (4) has a bounded solution $P(s)$ and let $S(s)$ be given by (5). It is not difficult to see that $S, S^* \in C_b([0,\infty), L(H,U))$.

Definition 5. *A bounded solution of (4) is called stabilizing for $\{A; G_i\}$ if $\{A + BS; G_i + H_iS\}$ is uniformly exponentially stable, where $S(t)$ is given by (5).*

Proposition 1. *(see* [5]*)The Riccati equation (4) has at most a bounded solution, which is stabilizing for $\{A; G_i\}$.*

Proof. If P and P_1 are two bounded solutions of (4) and P_1 is stabilizing for $\{A; G_i\}$ then $\Delta = P - P_1$ is a solution of (9). As in the proof of the above theorem, we get

$$\Delta(s)x = U^*_{\widehat{A}_1}(T,s)\Delta(T)U_{\widehat{A}_1}(T,s)x + \int_s^T U^*_{\widehat{A}_1}(r,s)[\sum_{i=1}^m G_i^*(r)\Delta(r)G_i(r)+$$

$$[(S_1 - S)^* \mathcal{K}(P)(S_1 - S)](r)U_{\widehat{A}_1}(r,s)xdr,$$

where $U_{\widehat{A}_1}(t,s)$ is the evolution operator generated by $\widehat{A}_1$. From the uniform exponential stability of $\{A+BS; G_i+H_iS\}$ it follows that $U_{\widehat{A}_1}(t,s)$ is uniformly exponentially stable. Since it exists $m_1 \in \mathbf{R}_+$ such that $\|(S_1 - S)(r)\| < m_1 \|\Delta(r)\|$, we use Gronwall's inequality to deduce that there exists $M, a > 0$, such that $\|\Delta(s)\| \leq Me^{-a(T-s)}$. As $T \to \infty$ we obtain $\|\Delta(s)\| = 0$, for all $s \in [0,\infty)$. The conclusion follows. $\square$

Reasoning as in [5], see Theorem 3.1 and stochasticize the proof we obtain the following result.

Proposition 2. *If $\{A; G_i\}$ is stabilizable then there exists a nonnegative boun- ded solution of the Riccati equation (4).*

Arguing as in [12] we can prove the following result:

Theorem 2. *Assume that $\{A, G_i; C\}$ is uniformly observable. If $P(t)$ is a nonnegative bounded solution of (4) then*

a) there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in \mathbf{R}_+$ (P is uniformly positive on $\mathbf{R}_+$);

b) P is a stabilizing solution (for $\{A; G_i\}$).

The next theorem is a consequence of the above theorem and of Proposition 2.

Theorem 3. *Assume $\{A, G_i; B\}$ is stabilizable and $\{A, G_i; C\}$ is uniformly observable. Then the Riccati equation (4) has a unique nonnegative bounded on $\mathbf{R}_+$ solution $P(t)$, which is a stabilizing solution and there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0,\infty)$.*

Proposition 3. *Assume that the hypotheses of the above theorem hod. If P is the unique and bounded on $\mathbf{R}_+$ solution of the Riccati equation (4) and S is the operator* *given by (5), then the equation*

$$g'(t) = -\left(A^* + S^*B^*\right) g(t) + C^*(t)C(t)r(t) \tag{10}$$

has a unique solution in $C_b([0,\infty), H)$*, where we consider the weak differentiability. Moreover, the function* $(t,x) \to \langle g'(t), x\rangle$ *is continuous on* $[0,\infty) \times H$.

Proof. Since $A + BS$ is the generator of an evolution operator $U_{A,B}(t,s)$, it is not difficult to see that the integral $g(s) = \int\limits_s^\infty U^*_{A,B}(\sigma,s)C^*(\sigma)C(\sigma)r(\sigma)d\sigma$ is convergent in H and $g(s)$ is bounded on $\mathbf{R}_+$. Differentiating the function $t \to \langle g(t), y\rangle\,, y \in H$, we see that $\frac{\partial}{\partial t}\langle g(t), y\rangle = \langle -\left(A^* + S^*B^*\right) g(t) + C^*(t)C(t)r(t), y\rangle$ and $g(t)$ is a solution of (10). If h is an other bounded solution then $(h-g)'(t) = -\left(A^* + S^*B^*\right)(h-g)(t)$. The unique solution of the last equation with the final condition $(h-g)(t) = h(t) - g(t)$ is $(h-g)(s) = U^*_{A,B}(t,s)\left[h(t) - g(t)\right]$. As $t \to \infty$ and since $U_{A,B}(t,s)$ is exponentially stable and the functions g and h are bounded on $\mathbf{R}_+$, we deduce that $(h-g)(s) = 0$, for all $s \geq 0$. Thus $h \equiv g$, and (10) has a unique solution. The last statement follows from the hypothesis, if we see that $\langle g'(t), x\rangle = -\langle g(t), (A(t) + B(t)S(t))\,x\rangle + \langle C^*(t)C(t)r(t), x\rangle$. □

We take the set of admissible controls $U_{ad} = \{u$ is an U- valued random variable, F_s- measurable such as $\overline{\lim\limits_{t\to\infty}} \frac{1}{t-s} E \int\limits_s^t \|u(\sigma)\|^2 d\sigma < \infty$ and $\sup\limits_{t\geq s} E\,\|x(t)\|^2 < \infty$, where x is the solution of (1)$\}$.

Theorem 4. *Assume that the hypotheses of the Theorem 3 hold. If P is the unique and bounded on $\mathbf{R}_+$ solution of the Riccati equation and $g(t)$ is the unique solution of (10) then the optimal control is*

$$u(t) = -\left[\mathcal{K}(P)(\sigma)\right]^{-1}\left[\mathcal{B}(P)(\sigma)x(\sigma) + B^*(\sigma)g(\sigma)\right]$$

and the optimal cost is

$$J(s) = \inf_{u\in U_{ad}} J(s,u) =$$

$$\overline{\lim_{t\to\infty}} \frac{1}{t-s}\left[\int_s^t \|C(\sigma)r(\sigma)\|^2 d\sigma - \int_s^t ||\left[\mathcal{K}(P)(\sigma)\right]^{-1/2} B^*(\sigma)g(\sigma)||^2 d\sigma\right].$$

Proof. We consider the function $F(t,x) = \langle P(t)x, x\rangle + 2\langle g(t), x\rangle$, which is continuous together its partial derivatives F_t, F_x, F_{xx} on $[0,\infty)\times H$, according the Remark 1 and the above proposition. Let $u \in U_{ad}$ and x be its response. Using Ito's formula for $F(t,x)$ and the strong solution of (1) we get

$$E\langle P(t)x(t), x(t)\rangle + 2\langle g(t), x(t)\rangle - E\langle P(s)x, x\rangle - 2\langle g(s), x\rangle =$$
$$-\int_s^t \|C(\sigma)[x(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle\, d\sigma +$$
$$\int_s^t \left\|\mathcal{K}(P)(\sigma)^{1/2}\left[u(\sigma) + [\mathcal{K}(P)(\sigma)]^{-1}[\mathcal{B}(P)(\sigma)x(\sigma) + B^*(\sigma)g(\sigma)]\right]\right\|^2$$
$$+\int_s^t \|C(\sigma)r(\sigma)\|^2\, d\sigma - \int_s^t \|\,[\mathcal{K}(P)(\sigma)]^{-1/2} B^*(\sigma)g(\sigma)\|^2 d\sigma.$$ Since $P(t)$ and $g(t)$ are bounded on $\mathbf{R}_+$ we multiply the last relation with $\frac{1}{t-s}$ and passing to the limit as $t\to\infty$ and, then, to the infimum we get the conclusion. □

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