STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 4, December 2005

PARTIAL SUMS OF CERTAIN MEROMORPHIC P-VALENT FUNCTIONS

B.A. FRASIN AND G. MURUGUSUNDARAMOORTHY

Abstract. In this paper, we study the ratio of meromorphic p-valent functions in the punctured disk $\mathcal{D} = \{z : 0 < |z| < 1\}$ of the form $f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} z^{p+k-1}$ to its sequence of partial sums of the form $f_n(z) = \frac{1}{z^p} + \sum_{k=1}^{n} a_{p+k-1} z^{p+k-1}$. Also, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$.

1. Introduction and definitions

Let Σ_p denotes the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} z^{p+k-1} \qquad (p \in \mathbb{N}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma_p$ is said to be in the class $\Sigma^*(p,\alpha)$ of meromorphic p-valently starlike functions of order α in $\mathcal{D}$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{2}$$

Furthermore, a function $f \in \Sigma_p$ is said to be in the class $\Sigma_{\mathcal{K}}(p,\alpha)$ of meromorphic p-valently convex functions of order α in $\mathcal{D}$ if and only if

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{3}$$

Received by the editors: 10.05.2005.

2000 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Meromorphic p-valent functions, meromorphic p-valently starlike and meromorphic p-valently convex functions, partial sums.

The class $\Sigma^*(p,\alpha)$ and various other subclasses of Σ_p have been studied rather extensively by Aouf *et.al.* [1-3], Joshi and Srivastava [6], Kulkarni *et. al.* [7], Mogra [8], Owa *et. al.* [9], Srivastava and Owa [11], Uralegaddi and Somantha [12], and Yang [13].

Let $\Omega_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{4}$$

And let $\Lambda_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\mathrm{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} < \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{5}$$

The classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$ were introduced and studied by the authors [5].

In [5] the authors obtained the following sufficient conditions for a function of the form (1.1) to be in the classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$.

Lemma 1. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{k=1}^{\infty}(p+k+\delta-1+|p+k+2\alpha-\delta-1|)a_{p+k-1} < 2(p-\alpha). \tag{6}$$

for some $\alpha(0 \le \alpha < p)$ *and some* $\delta(\alpha < \delta \le p)$, *then* $f(z) \in \Omega_p(\alpha)$.

Lemma 2. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{k=1}^{\infty}(p+k-1)(p+k+\delta-1+|p+k+2\alpha-\delta-1|)a_{p+k-1} < 2(p-\alpha) \tag{7}$$

for some $\alpha(0 \le \alpha < p)$ *and some* $\delta(\alpha < \delta \le p)$, *then* $f(z) \in \Lambda_p(\alpha)$.

In view of Lemma 1 and Lemma 2, we now define the subclasses $\Omega_p^*(\alpha) \subset \Omega_p(\alpha)$ and $\Lambda_p^*(\alpha) \subset \Lambda_p(\alpha)$, which consist of functions $f(z) \in \Sigma_p$ satisfying the conditions (1.6) and (1.7), respectively.(see [5]).

In the present paper, and by following the earlier work of Silverman [10] (see also [4]), we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_n(z) = \frac{1}{z^p} + \sum_{k=1}^{n} a_{p+k-1} z^{p+k-1} \qquad (p \in \mathbb{N}), \tag{8}$$

when the coefficients of $f(z)$ are satisfy the condition (1.6) or (1.7). More precisely, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f^{'}(z)\right\}$.

For the notational convenience we shall henceforth denote

$$\sigma_k(p,\delta,\alpha) := p+k+\delta-1+|p+k+2\alpha-\delta-1| \tag{9}$$

2. Main results

Theorem 1. *If $f(z)$ of the form (1.1) satisfies the condition (1.6), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} \qquad (z\in\mathcal{U}) \tag{1}$$

The results (2.1) is sharp for every n, with extermal function

$$f(z) = \frac{1}{z} + \frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} z^{p+n} \qquad (n\geq 0). \tag{2}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\left[\frac{f(z)}{f_n(z)} - \left(\frac{\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\right)\right]$$

$$= \frac{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p} + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p}} \tag{3}$$

It suffices to show that $|w(z)|\leq 1$. Now, from (2.3) we can write

$$w(z) = \frac{\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{2+2\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p} + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}$$

to find that

$$|w(z)| \leq \frac{\frac{\sigma_{k+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} |a_{p+k-1}|}{2-2\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+1} - \frac{\sigma_{k+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+1}}.$$

Now $|w(z)|\leq 1$ if

$$2\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 2-2\sum_{k=1}^{n}|a_{p+k-1}|,$$

which is equivalent to

$$\sum_{k=1}^{n}|a_{p+k-1}|+\sum_{k=n+1}^{\infty}\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)|a_{p+k-1}|\leq 1.$$

From the condition (1.6), it is sufficient to show that

$$\sum_{k=1}^{n}|a_{p+k-1}|+\sum_{k=n+1}^{\infty}\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)|a_{p+k-1}|\leq\sum_{k=1}^{\infty}\frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)}|a_{p+k-1}| \qquad (4)$$

which is equivalent to

$$\sum_{k=1}^{n}\frac{\sigma_k(p,\delta,\alpha)-2(p-\alpha)}{2(p-\alpha)}|a_{p+k-1}|+\sum_{k=n+1}^{\infty}\frac{\sigma_k(p,\delta,\alpha)-\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}|a_{p+k-1}|\geq 0.$$

To see that the function given by (2.2) gives the sharp result, we observe that for $z=re^{\pi i/(n+p+1)}$

$$\begin{aligned}\frac{f(z)}{f_n(z)} &= 1+\frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}z^{n+p+1}\to 1-\frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\\ &= \frac{\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\quad\text{when } r\to 1^{-}.\end{aligned}$$

Therefore we complete the proof of Theorem 1.

Theorem 2. *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}\geq\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)}\qquad(z\in\mathcal{U}). \qquad (5)$$

The results (2.5) is sharp for every n, with extremal function

$$f(z)=\frac{1}{z}+\frac{2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)}z^{p+n}\qquad(n\geq 0). \qquad (6)$$

Proof. We write

$$\frac{1+w(z)}{1-w(z)}=\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\left[\frac{f(z)}{f_n(z)}-\left(\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)}\right)\right]$$

$$=\frac{1+\sum_{k=1}^{n}a_{p+k-1}z^{k+p}+\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum_{k=n+1}^{\infty}a_{p+k-1}z^{k+p}}{1+\sum_{k=1}^{n}a_{p+k-1}z^{k+p}}$$

where

$$w(z)=\frac{\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}{2+2\sum\limits_{k=2}^{n}|a_{p+k-1}|+\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}.$$

Now

$$|w(z)|\le\frac{\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}{2-2\sum\limits_{k=1}^{n}|a_{p+k-1}|-\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}\le 1$$

if

$$\sum_{k=1}^{n}|a_{p+k-1}|+\sum_{k=n+1}^{\infty}\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}|a_{p+k-1}|\le 1. \qquad (7)$$

The left hand side of (2.7) is bounded above by
$\sum\limits_{k=1}^{\infty}[(p+k-1)\sigma_k(p,\delta,\alpha)/(2(p-\alpha))]\,|a_{p+k-1}|$ if

$$\begin{aligned}&\frac{1}{2(p-\alpha)}\sum_{k=1}^{n}[(p+k-1)\sigma_k(p,\delta,\alpha)-2(p-\alpha)]\,|a_{p+k-1}|\\&+\sum_{k=n+1}^{\infty}[(p+k-1)\sigma_k(p,\delta,\alpha)-(p+n)\sigma_{n+1}(p,\delta,\alpha)]\,|a_{p+k-1}|\\ \ge\ & 0,\end{aligned}$$

and the proof is complete.

We next determine bounds for $f_n(z)/f(z)$.

Theorem 3. (a) *If $f(z)$ of the form (1.1) satisfies the condition (1.6), then*

$$\mathrm{Re}\left\{\frac{f_n(z)}{f(z)}\right\}\ge\frac{\sigma_{n+1}(p,\delta,\alpha)}{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}\qquad(z\in\mathcal{U}). \qquad (8)$$

(b) *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then*

$$\mathrm{Re}\left\{\frac{f_n(z)}{f(z)}\right\}\ge\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}\qquad(z\in\mathcal{U}). \qquad (9)$$

The results (2.8) and(2.9) are sharp for the functions given by (2.2) and (2.6), respectively.

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}{2(p-\alpha)}\left[\frac{f_n(z)}{f(z)} - \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}\right)\right]$$

$$= \frac{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p} + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{1+\sum\limits_{k=1}^{\infty} a_{p+k-1}z^{k+p}}$$

where

$$|w(z)| \leq \frac{\frac{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}{2-2\sum\limits_{k=1}^{n}|a_{p+k-1}| - \left(\frac{2(p-\alpha)-\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=1}^{n}|a_{p+k-1}| + \sum_{k=n+1}^{\infty}\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}|a_{p+k-1}| \leq 1. \tag{10}$$

The left hand side of (2.10) is bounded above by $\sum\limits_{k=1}^{\infty}[\sigma_k(p,\delta,\alpha)/(2(p-\alpha))]\,|a_{p+k-1}|$, the proof is completed.

We next turn to ratios involving derivatives

Theorem 4.*If $f(z)$ of the form (1.1) satisfies the condition (1.6), then for $z\in\mathcal{U}$,*

(a) $\operatorname{Re}\{f'(z)/f_n'(z)\} \geq [\sigma_{n+1}(p,\delta,\alpha)-2(n+1)(p-\alpha)]/\sigma_{n+1}(p,\delta,\alpha)$.

(b) $\operatorname{Re}\{f_n'(z)/f'(z)\} \geq \sigma_{n+1}(p,\delta,\alpha)/[\sigma_{n+1}(p,\delta,\alpha)+2(n+1)(p-\alpha)]$.

*The results in (*a*) and in (*b*) are sharp with the function given by (2.2)*

Proof. We prove only (a), which is similar to the proof of Theorem 1. The proof of (b) follows the pattern of that in Theorem 3(a). We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\left[\frac{f'(z)}{f_n'(z)} - \left(\frac{\sigma_{n+1}(p,\delta,\alpha)-2(n+1)(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\right)\right]$$

where

$$w(z) = \frac{\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_{p+k-1}z^{k+p}}{2+2\sum\limits_{k=2}^{n} k a_{p+k-1}z^{k+p} + \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty} k a_{p+k-1}z^{k+p}}$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{n} k\,|a_{p+k-1}| + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)} \sum_{k=n+1}^{\infty} k\,|a_{p+k-1}| \leq 1.$$

From the condition (1.6), it is sufficient to show that

$$\sum_{k=2}^{n} k\,|a_{p+k-1}| + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)} \sum_{k=n+1}^{\infty} k\,|a_{p+k-1}| \leq \sum_{k=2}^{\infty} \frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)}\,|a_{p+k-1}|$$

which is equivalent to

$$\sum_{k=2}^{n} \left(\frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)} - k\right)|a_{p+k-1}| + \sum_{k=n+1}^{\infty} \left(\frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)} - \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}k\right)|a_{p+k-1}| \geq 0,$$

and the proof is complete.

Theorem 5. *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then for $z \in \mathcal{U}$,*

(a) $\operatorname{Re}\{f'(z)/f_n'(z)\} \geq [(p+n)\sigma_{n+1}(p,\delta,\alpha) - 2(p-\alpha)(n+1)]/[(p+n)\sigma_{n+1}(p,\delta,\alpha)]$.

(b) $\operatorname{Re}\{f_n'(z)/f'(z)\} \geq [(p+n)\sigma_{n+1}(p,\delta,\alpha)]/[(p+n)\sigma_{n+1}(p,\delta,\alpha) + 2(p-\alpha)(n+1)]$.

*The results in (*a*) and in (*b*) are sharp with the function given by (2.6).*

Proof. It is well known that $f \in \Lambda_p(\alpha) \Leftrightarrow zf' \in \Omega_p(\alpha)$. In particular, f satisfies condition (1.7) if and only if zf' satisfies condition (1.6). Thus, (a) is an immediate consequence of Theorem 1 and (b) follows directly from Theorem 3(a).

References

[1] Aouf, M.K., *New criteria for multivalent meromorphic starlike functions of order alpha*, Proc. Japan. Acad. Ser. A. Math. Sci. 69 (1993), 66-70.

[2] Aouf, M.K., Hossen, H.M., *New criteria for meromorphic p-valent starlike functions*, Tsukuba J. Math. 17 (1993), 481-486.

[3] Aouf, M.K., Srivastava, H.M., *A new criteria for meromorphic p-valent convex functions of order alpha*, Math. Sci. Res. Hot-line 1 (8) (1997), 7-12.

[4] Cho, N.E., Owa, S., *Partial sums of certain meromorphic functions*, JIPAM 5 (2) (2004), 1-7.

[5] Frasin, B.A., Murugusundaramoorthy, G., *New subclasses of meromorphic p-valent functions*, Submitted..

[6] Joshi, S.B., Srivastava, H.M., *A certain family of meromorphically multivalent functions*, Computers Math. Appl. 38 (3) (1999), 201-211.

[7] Kukarni, S.R., Naik, U.H., Srivastava, H.M., *A certain class of meromorphically p-valent quasi-convex functions,* Pan Amer. math. J. 8 (1) (1998), 57-64.

[8] Mogra, M.L., *Meromorphic multivalent functions with positive coefficients I and II*, Math. Japon. 35 (1990), 1-11 and 1089-1098.

[9] Owa, S., Darwish, H.E., Aouf, M.K., *Meromorphic multivalent functions with positive and fixed second coefficients*, Math. Japon. 46 (1997), 231-236.

[10] Silverman, H., *Partial sums of starlike and convex functions,* J.Math Anal.&.Appl. 209 (1997), 221-227.

[11] Srivastava, H.M., Owa, S., (Eds.),"*Current Topics in Analytic Function Theory*," World Scientific, Singapore/New Jersey/London/Hong Kong, (1992).

[12] Uralegaddi, B.A., Somanatha, C., *Certain classes of meromorphic multivalent functions,* Tamkang J. Math. 23 (1992), 223-231.

[13] Yang, D.G., *On new subclasses of meromorphic p-valent functions*, J. Math. Res. Exposition 15 (1995), 7-13.

Department of Mathematics, Al al-Bayt University, P.O. Box: 130095, Mafraq, Jordan
E-mail address: `bafrasin@yahoo.com`

Department of Mathematics, Vellore Institute of Technology, Deemed University, Vellore, TN-632 014, India
E-mail address: `gmsmoorthy@yahoo.com`