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AN APPLICATION OF MACKEY'S SELECTION LEMMA

MĂDĂLINA ROXANA BUNECI

Abstract. Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. Let us denote by d_F the restriction of the domain map to G^F and by r' the restriction of the range map to the isotropy group bundle of G. We shall prove that if d_F is open, then r' is open and d_F has a regular Borel cross section. Conversely, we shall prove that if r' is open and d_F admits a regular cross section (a right inverse which carries each compact subset of $G^{(0)}$ into a relatively compact subset of G^F), then d_F is open. We shall also prove that, if d_F is open, then F is a closed subset of $G^{(0)}$, and the orbit space $G^{(0)}/G$ is a proper space. If F is closed and regular (the intersection of F with the saturated of any compact subset of $G^{(0)}$ is relatively compact) and $G^{(0)}/G$ is proper, then d_F is open.

1. **Introduction**

We shall consider a locally compact groupoid G and a set F containing exactly one element from each orbit of G. We shall study the connection between the openness of d_F, the restriction of the domain map to G^F, and the existence of a regular cross section of d_F (a right inverse which carries each compact subset of $G^{(0)}$ into a relatively compact subset of G^F). The motivation for studying the map d_F comes from the fact that if F is closed and d_F is open, then G and G_F^F are (Morita) equivalent locally compact groupoids (in the sense of Definition 2.1/p. 6 [4]). A result of Paul Muhly, Jean Renault and Dana Williams states that the C^*-algebras associated to (Morita)

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equivalent locally compact second countable groupoids are strongly Morita equivalent (Theorem 2.8/p. 10 [4]). Also the notion of topological amenability is invariant under the equivalence of groupoids (Theorem 2.2.7/p. 50 [1]). Consequently, if F is closed and d_F is open, then G and the bundle group G_F^F have strongly Morita equivalent C^*-algebras. Also, the equivalence of G and G_F^F implies that G is amenable if and only if each isotropy group G_u^u is amenable.

For establishing notation, we include some definitions that can be found in several places (e.g. [5]). A groupoid is a set G, together with a distinguished subset $G^{(2)} \subset G \times G$ (called the set of composable pairs), and two maps:

$$\begin{aligned} (x,y) &\rightarrow xy \left[: G^{(2)} \rightarrow G\right] \text{ (product map)} \\ x &\rightarrow x^{-1} \; [: G \rightarrow G] \text{ (inverse map)} \end{aligned}$$

such that the following relations are satisfied:

(1) If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)\,z = x\,(yz)$.

(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.

(3) For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)\,x^{-1} = z$.

(4) For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}\,(xy) = y$.

The maps r and d on G, defined by the formulae $r\,(x) = xx^{-1}$ and $d\,(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd\,(x) = r\,(x)\,x = x$. It is useful to note that a pair (x,y) lies in $G^{(2)}$ precisely when $d\,(x) = r\,(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibers of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. G_A^A becomes a groupoid (called the reduction of G to A) with the unit space A, if we define $\left(G_A^A\right)^{(2)} = G^{(2)} \cap \left(G_A^A \times G_A^A\right)$. For each unit u, G_u^u is a group, called isotropy group at u. The group bundle

$$\{x \in G : r(x) = d(x)\}$$

is denoted G', and is called the isotropy group bundle of G. The relation $u\tilde{\ }v$ iff $G_v^u \neq \phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted $[u]$. Let

$$R = (r, d)(G) = \{(r(x), d(x)), \ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. We are exclusively concerned with topological groupoids which are locally compact Hausdorff. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets.

2. Necessary and sufficient conditions for the openness of d_F.

Definition 1. *Let X, Y be two topological spaces. A cross section of a map $f : X \to Y$ is a function $\sigma : Y \to X$ such that $f(\sigma(y)) = y$ for all $y \in Y$. We shall say that the cross section σ is regular if $\sigma(K)$ has compact closure in X for each compact set K in Y.*

We shall need the following lemma proved by Mackey (Lemma 1.1/p. 102 [3]):

Lemma 1. *If X and Y are second countable, locally compact spaces, and $f : X \to Y$ is a continuous open function onto Y, then f has a Borel regular cross section.*

Proposition 1. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. Let us define the function $e : G^{(0)} \to G^{(0)}$ by*

$$e(u) = F \cap [u], \ u \in G^{(0)}$$

If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then the function e is continuous and F is a closed subset of $G^{(0)}$.

Proof. Let $(u_i)_i$ be a net converging to u in $G^{(0)}$. Let $x \in G$ be such that $r(x) = e(u)$ and $d(x) = u$. Since $(u_i)_i$ converges to $d_F(x)$ and d_F is an open map, we may pass to a subnet and assume that there is a net $(x_i)_i$ converging to x in G^F such that $d_F(x_i) = u_i$. It is easy to see that $r(x_i) = e(u_i)$ ($r(x_i) \in F$ and $r(x_i) \in [d(x_i)] = [u_i]$). Thus $e(u_i) = r(x_i)$ converges to $r(x) = e(u)$. Since $G^{(0)}$ is Hausdorff, F is closed in $G^{(0)}$, being the image of the map e whose square is itself. □

Proposition 2. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then graph*

$$R = \{(r(x), d(x)),\ x \in G\}$$

of the equivalence relation induced on $G^{(0)}$ is closed in $G^{(0)} \times G^{(0)}$, and the map $(r,d) : G \to R$, $(r,d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$.

Proof. Let us define the function $e : G^{(0)} \to G^{(0)}$ by

$$e(u) = F \cap [u],\ u \in G^{(0)}.$$

By Proposition 1, the function e is continuous. Let $((u_i, v_i))_i$ be a net in R which converges to (u, v) in $G^{(0)} \times G^{(0)}$ (with respect to with the product topology). Then $(u_i)_i$ converges to u, $(v_i)_i$ converges to v, and $u_i \sim v_i$ for all i. We have

$$\begin{aligned} \lim_i e(u_i) &= e(u) \\ \lim_i e(v_i) &= e(v) \end{aligned}$$

because e is continuous. On the other hand, the fact that $u_i \sim v_i$ for all i implies that $e(u_i) = e(v_i)$ for all i. Hence $e(u) = e(v)$, or equivalently, $u \sim v$. Therefore $(u, v) \in R$.

Let us prove that the map $(r,d) : G \to R$, $(r,d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Let $x \in G$, and let $((u_i, v_i))_i$ be a net in R converging to $(r,d)(x)$. Then $(u_i)_i$ converges to $r(x)$, $(v_i)_i$ converges to $d(x)$, and $u_i \sim v_i$ for all i. Let $s \in G$ be such that $r(s) = e(r(x))$

and $d(s) = r(x)$ and let $t = sx$. Obviously, $s, t \in G^F$ and

$$\begin{aligned}\lim_i u_i &= r(x) = d(s) \\ \lim_i v_i &= d(x) = d(sx) = d(t).\end{aligned}$$

Since d_F is an open map, we may pass to subnets and assume that there is a net $(s_i)_i$ converging to s in G^F and there is a net $(t_i)_i$ converging to t in G^F such that $d_F(s_i) = u_i$ and $d_F(t_i) = v_i$. The fact that $e(u_i)$ is the only element of F, which is equivalent to $u_i \tilde{} \ v_i$, implies that $r(s_i) = e(u_i) = e(v_i) = r(t_i)$. We have

$$\begin{aligned}\lim_i s_i^{-1} t_i &= s^{-1} t = x \\ r\left(s_i^{-1} t_i\right) &= d(s_i) = u_i,\ d\left(s_i^{-1} t_i\right) = d(t_i) = v_i\end{aligned}$$

Therefore the map (r, d) is open □

Corollary 1. *Let G be a locally compact groupoid having open range map. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then the orbit space $G^{(0)}/G$ is proper.*

Proof. The fact that $G^{(0)}/G$ is a proper space means that $G^{(0)}/G$ is Hausdorff and the map $(r, d) : G \to R$, $(r, d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Let us note that the quotient map $\pi : G^{(0)} \to G^{(0)}/G$ is open (because the range map of G is open). Since the graph R of the equivalence relation is closed in $G^{(0)} \times G^{(0)}$, it follows that $G^{(0)}/G$ is Hausdorff. □

Lemma 2. *Let G be a locally compact groupoid having open range map. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then F and $G^{(0)}/G$ are homeomorphic spaces.*

Proof. Let $\pi : G^{(0)} \to G^{(0)}/G$ be the quotient map. We prove that the map $\pi_F : F \to G^{(0)}/G, \pi_F(x) = \pi(x)$ is a homeomorphism. It suffices to prove that π_F is an open map (because π_F is one-to-one from F onto $G^{(0)}/G$). Let $u \in F$ and $(\dot{u}_i)_i$ be a net converging to $\pi(u)$ in $G^{(0)}/G$. Since $\pi \circ d_F$ is open, we may pass to a subnet and assume that there is a net $(x_i)_i$ converging to u in G^F such that $\pi(d_F(x_i)) = \dot{u}_i$. Then $(r(x_i))_i$ is a net in F which converges to u. □

Remark 1. *Let G be a locally compact groupoid. If the map $(r,d) : G \to R$ is open (where $R = \{(r(x), d(x)), x \in G\}$ is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$), then the map $r' : G' \to G^{(0)}, r'(x) = r(x)$, is open, where*

$$G' = \{x \in G : r(x) = d(x)\},$$

is the isotropy group bundle of G.

Proposition 3. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then d_F has Borel regular cross section.*

Proof. If d_F is open, then according to Proposition 1, F is a closed subset of $G^{(0)}$. Therefore G^F is a locally compact space, and we may apply Lemma 1. □

We shall need a system of measures

$$\{\beta_v^u, (u,v) \in (r,d)(G)\}$$

satisfying the following conditions:

1. $supp(\beta_v^u) = G_v^u$ for all $u\,\tilde{}\,v$.
2. $\sup\limits_{u,v} \beta_v^u(K) < \infty$ for all compact $K \subset G$.
3. $\int f(y)\, d\beta_v^{r(x)}(y) = \int f(xy)\, d\beta_v^{d(x)}(y)$ for all $x \in G$ and $v\,\tilde{}\,r(x)$.

In Section 1 of [6] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally support, which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1. Renault defines $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)\, d\beta_v^v(y)$ as usual). If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$. We obtain another construction of a system a measures with the above properties if in the proof of Theorem 8/p. 331[2] we replace the regular cross section of $G^u \overset{d}{\to} G^{(0)}$ (in the transitive case) with a regular cross section of $G^F \overset{d}{\to} G^{(0)}$, where F is a subset of $G^{(0)}$ meeting each orbit exactly once.

Lemma 3. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once and let us denote $e(u)$ the unique element of F equivalent to u. If the map*

$$u \mapsto \int f(y)\, d\beta_u^{e(u)}(y) \;\left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous for any continuous function with compact support, $f : G \to \mathbf{C}$, then the map

$$d_F : G^F \to G^{(0)},\ d_F(x) = d(x)\,.$$

is open.

Proof. Let $x_0 \in G^F$ and let U be a nonempty compact neighborhood of x_0. Choose a nonnegative continuous function, f on G , with $f(x_0) > 0$ and $supp\,(f) \subset U$. Let W be the set of units u with the property that $\beta_u^{e(u)}(f) > 0$. Then W is an open neighborhood of $u_0 = d(x_0)$ contained in $d_F(U)$. □

Proposition 4. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ containing exactly one element from each orbit of G, and let us denote $e(u)$ the unique element of F equivalent to u. Let us assume that the map $r' : G' \to G^{(0)}, r'(x) = r(x)$ is open, where G' is the isotropy group bundle of G. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, has a regular cross section σ, then for each continuous with compact support function $f : G \to \mathbf{C}$, the map*

$$u \to \int f(y)\, d\beta_u^{e(u)}(y)$$

is continuous on G.

Proof. By Lemma 1.3/p. 6 [6], for each $f : G \to \mathbf{C}$ continuous with compact support, the function $u \to \int f(y)\, d\beta_u^u(y)\ \left[: G^{(0)} \to \mathbf{C}\right]$ is continuous. Let $(u_i)_i$ be a sequence in $G^{(0)}$ converging to u. Let $x_i = \sigma(u_i)^{-1}$. Since σ is regular, it follows that $(x_i)_i$ has a convergent subsequence in G^F. Let x be the limit of this subsequence. Let $f : G \to \mathbf{C}$ be a continuous function with compact support and let g be a continuous extension on G of $y \to f(xy)\left[: G^{d(x)} \to \mathbf{C}\right]$. Let K be the compact set $\left(\{x, x_i, i = 1, 2, ..\}^{-1} supp(f) \cup supp(g)\right) \cap r^{-1}(\{d(x), d(x_i), i = 1, 2, ...\})$. We have

$$\begin{aligned}
& \left|\int f(y)\, d\beta_u^{e(u)}(y) - \int f(y)\, d\beta_{u_i}^{e(u_i)}(y)\right| \\
= & \left|\int f(xy)\, d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
= & \left|\int g(y)\, d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
\leq & \left|\int g(y)\, d\beta_{d(x)}^{d(x)}(y) - \int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y)\right| + \\
& + \left|\int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y)\right| \\
\leq & \left|\int g(y)\, d\beta_u^u(y) - \int g(y)\, d\beta_{u_i}^{u_i}(y)\right| + \\
& + \sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)|\, \beta_{u_i}^{u_i}(K)
\end{aligned}$$

A compactness argument shows that $\sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)|$ converges to 0. Also $\left|\int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y)\right|$ converges to 0, because the function $u \to \int f(y)\, d\beta_u^u(y)$ is continuous on $G^{(0)}$. Hence

$$\left|\int f(y)\, d\beta_u^{e(u)}(y) - \int f(y)\, d\beta_{u_i}^{e(u_i)}(y)\right|$$

converges to 0. □

Corollary 2. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the restriction r' of the range map to the isotropy group bundle G' of G is open, and if the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, has a regular cross section, then d_F is an open map.*

Theorem 1. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once, and let $d_F : G^F \to G^{(0)}$ be the map defined by $d_F(x) = d(x)$ for all $x \in G^F$. If d_F is open then d_F admits a Borel regular cross section. If the restriction r' of the range map to the isotropy group bundle G' of G is open and if d_F admits a regular cross section, then d_F is an open map.*

Proof. If d_F is an open map, then, according Proposition 3, d_F has a regular cross section. Conversely, if d_F admits a regular cross section, then applying Proposition 4 and Lemma 3, it follows that d_F is open. □

Remark 2. *Let us assume that $G^{(0)}/G$ is* ***proper****. There is a regular Borel cross section σ_0 of the quotient map $\pi : G^{(0)} \to G^{(0)}/G$. Let us assume that $F = \sigma_0 \left(G^{(0)}/G\right)$ is* ***closed*** *in $G^{(0)}$. Then the function $e : G^{(0)} \to G^{(0)}$ defined by $e(u) = F \cap [u]$ is continuous. If $\sigma_1 : R \to G$ is regular Borel cross section of (r, d), then $\sigma : G^{(0)} \to G^F$, $\sigma(u) = \sigma_1(e(u), u)$ is a Borel regular cross section of d_F. Therefore is that case d_F is open.*

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University Constantin Brâncuşi of Târgu-Jiu, Bulevardul Republicii, Nr. 1, 210152 Târgu-Jiu, Romania

E-mail address: ada@utgjiu.ro