STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 3, September 2005

CYCLIC REPRESENTATIONS AND PERIODIC POINTS

GABRIELA PETRUŞEL

Abstract. The purpose of this note is to give some existence results of periodic points for some classes of single-valued operators. The fixed point structures technique and an abstract periodic point lemma given by I. A. Rus are used.

1. Introduction

Throughout this paper, we will use the notations and terminologies in [4], [5].

Let (X, d) be a metric space and $f : X \to X$ an operator. By $F_f := \{x \in X | \ x = f(x)\}$ we will denote the fixed point set of the operator f.

We will also use the following symbols:

$P(X) := \{Y \subseteq X | Y \neq \emptyset\}$, $P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed}\}$, $P_{cp}(X) := \{Y \in P(X) | \ Y \text{ is compact}\}$ and $P_b(X) := \{Y \in P(X) | \ Y \text{ is bounded }\}$.

Let X, Y be nonempty sets. We will denote by $\mathbb{M}(X, Y)$ the set of all single-valued operators from $f : X \to Y$. If $X = Y$ then $\mathbb{M}(Y) := \mathbb{M}(Y, Y)$.

Definition 1.1. Let X be a nonempty set. By definition (see [4]), the triple $(X, S(X), M)$ is a fixed point structure (briefly f. p. s.) if:

(i) $S(X) \subset P(X), S(X) \neq \emptyset$

(ii) $M : P(X) \to \bigcup_{Y \in P(X)} \mathbb{M}(Y)$ is a selection operator, such that if $Z \subset Y, Z \neq \emptyset$ then $M(Z) \supset \{f_{|Z} | f \in M(Y), Z \in I(f)\}$

(iii) for each $Y \in S(X)$ and $f \in M(Y)$ we have that $F_f \neq \emptyset$.

Received by the editors: 12.09.2005.

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Periodic point, fixed point, cyclic representation, fixed point structure.

Definition 1.2. (I. A. Rus [5]) Let X be a nonempty set and $f : X \to X$ an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ (where $X_i \subset X$, for each $i \in \{1, 2, \cdots, m\}$) is a cyclic representation of X with respect to f if $f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

In [3], W. A. Kirk, P. S. Srinivasan, P. Veeramani proved some fixed point theorems for single-valued operators satisfying some cyclical contractive assumptions. Then, I. A. Rus generalize these results in terms of the fixed point structures (see [5]).

Also, in Rus [5], the following periodic points lemma is given:

Lemma 1.3. *Let $(X, S(X), M)$ be a fixed point structure, where X is a nonempty set. Let $A_i \in P(X)$, for each $i \in \{1, 2, \cdots, m\}$. Denote $Y := \bigcup_{i=1}^{m} A_i$ and consider $f : Y \to Y$. Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) $A_i \in S(X)$ for some $i \in \{1, 2, \cdots, m\}$;

(iii) $g_1, g_2 \in M(Y)$ implies $g_1 \circ g_2 \in M(Y)$.

Then $F_{f^m} \neq \emptyset$.

The purpose of this paper is to give some applications of the previous lemma.

2. Periodic points for Knaster-Tarski type operators

Let $(X, \leq)$ be an ordered set,
$S(X) := \{Y \in P(X) | (Y, \leq) \text{ is a complete latice}\}$ and
$M(Y) := \{f : Y \to Y | \text{ f is increasing }\}$. Then $(X, S(X), M)$ is a f. p. s. (Knaster-Tarski, see [1]).

Then, by applying Lemma 1.3., one obtains:

Theorem 2.1. *Let $(X, \leq)$ be an ordered set, $A_i \in P(X)$, for $i \in \{1, 2, \cdots, m\}$, such that there is $i_0 \in \{1, 2, \cdots, m\}$ with A_{i_0} a complete lattice. Denote $Y := \bigcup_{i=1}^{m} A_i$ and consider $f : Y \to Y$. Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) $f(x_1) \leq f(x_2)$, *for each* $x_1 \in A_i$ *and each* $x_2 \in A_{i+1}$, $(i \in \{1, 2, \cdots, m\})$ *with* $x_1 \leq x_2$ *(where* $A_{m+1} = A_1$*).*

Then $F_{f^m} \neq \emptyset$.

Proof. Let us remark that the fixed point structure of Knaster-Tarski satisfies the conditions (i)-(iii) in Lemma 1.3. □

3. **Periodic points for generalized contractions**

Let (X, d) be a complete metric space. Then the operator $f : X \to X$ is called a φ-contraction if there exists a comparison function φ (i. e. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and $(\varphi^n(t))_{n\in\mathbb{N}} \to 0$, as $n \to +\infty$, for each $t > 0$) such that

$$d(f(x_1), f(x_2)) \leq \varphi(d(x_1, x_2)), \text{ for all } x_1, x_2 \in X.$$

If we consider $S(X) := P_{cl}(X)$ and one define

$M(Y) := \{f : Y \to Y |\text{exists a comparison function } \varphi \text{ such that f is a } \varphi\text{-contraction}\}$, then $(X, S(X), M)$ is a f. p. s. (see Rus [4])).

The following result follow now from Lemma 1. 3.

Theorem 3.1. *Let* (X, d) *be a complete metric space,* $A_i \in P(X)$, *for* $i \in \{1, 2, \cdots, m\}$, *such that there is* $i_0 \in \{1, 2, \cdots, m\}$ *with* $A_{i_0} \in P_{cl}(X)$. *Denote* $Y := \bigcup_{i=1}^{m} A_i$ *and consider* $f : Y \to Y$. *Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* f*;*

(ii) there exists a comparison function φ *such that*

$$d(f(x_1), f(x_2)) \leq \varphi(d(x_1, x_2)), \textit{ for all } x_1 \in A_i, \textit{ and } x_2 \in A_{i+1}, i \in \{1, 2, \cdots, m\},$$

where $A_{m+1} = A_1$.

Then $F_{f^m} \neq \emptyset$.

Proof. Let $g_1, g_2 \in M(Y)$. It follows that there exist the comparison functions φ_1, φ_2 such that g_i is a φ_i-contraction, for $i \in \{1, 2\}$. Since the composition of two comparison functions is a comparison function, it follows immediately that the condition (iii) in Lemma 1.3. holds. □

4. Periodic points for contractive operators

Let (X,d) be a metric space. Then the operator $f : X \to X$ is called contractive if $d(f(x_1), f(x_2)) < d(x_1, x_2)$, for all $x_1, x_2 \in X, x_1 \neq x_2$. If $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y| \text{ f is contractive }\}$. If (X,d) is a compact metric space, then $(X, S(X), M)$ is a f. p. s. (Nemytzki-Edelstein, see [4])).

From Lemma 1.3. we have:

Theorem 3.1. *Let (X,d) be a compact metric space, $A_i \in P(X)$, for $i \in \{1, 2, \cdots, m\}$, such that there is $i_0 \in \{1, 2, \cdots, m\}$ with $A_{i_0} \in P_{cl}(X)$. Denote $Y := \bigcup_{i=1}^{m} A_i$ and consider $f : Y \to Y$. Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) $d(f(x_1), f(x_2)) < d(x_1, x_2)$, for each $x_1 \in A_i$, and $x_2 \in A_{i+1}$, with $x_1 \neq x_2$, for $i \in \{1, 2, \cdots, m\}$, where $A_{m+1} = A_1$.

Then $F_{f^m} \neq \emptyset$.

Proof. Let g, h be contractive operators. Then, for any two elements x_1, x_2 from X, with $x_1 \neq x_2$, we have: $d((g \circ h)(x_1), (g \circ h)(x_2)) \leq d(h(x_1), h(x_2)) < d(x_1, x_2)$. hence all the conditions of Lemma 1.3. are satisfy. □

5. Periodic points for nonexpansive operators

Let (X,d) be an uniformly convex Banach space. Then the operator $f : X \to X$ is called nonexpansive if $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$, for all $x_1, x_2 \in X$. If $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y| \text{ f is nonexpansive }\}$. Then $(X, S(X), M)$ is a f. p. s.(Browder - Ghöde - Kirk, see [1], [2]).

For nonexpansive operators we have:

Theorem 4.1. *Let X be an uniformly convex Banach space, $A_i \in P(X)$, for $i \in \{1, 2, \cdots, m\}$, such that there is $i_0 \in \{1, 2, \cdots, m\}$ with $A_{i_0} \in P_{b,cl,cv}(X)$. Denote $Y := \bigcup_{i=1}^{m} A_i$ and consider $f : Y \to Y$. Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$, *for each* $x_1 \in A_i$, *and* $x_2 \in A_{i+1}$, *for* $i \in \{1, 2, \cdots, m\}$, *where* $A_{m+1} = A_1$.

Then $F_{f^m} \neq \emptyset$.

Proof. Since the composition of two nonexpansive operators is a nonexpansive operator, we remark that the condition (iii) in Lemma 1.3 holds. The conclusion follows now by Lemma 1.3. □

6. Periodic points for Perov type operators

Let (X, d) be a generalized metric space, in the sense that $d(x, y) \in \mathbb{R}^k$. The operator $f : X \to X$ is called a Perov type contraction (or S-contraction) if $S \in \mathcal{M}_{kk}(\mathbb{R})$, with $S^n \to 0$, as $n \to +\infty$, such that $d(f(x_1), f(x_2)) \leq S \cdot d(x_1, x_2))$, for all $x_1, x_2 \in X$. If $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y|$ f is a Perov contraction $\}$. Then $(X, S(X), M)$ is a f. p. s. (Perov, see [4])).

In the setting of the Perov's f. p. s., Lemma 1. 3. gives us:

Theorem 5. 1. *Let* (X, d) *be a complete generalized metric space,* $A_i \in P(X)$, *for* $i \in \{1, 2, \cdots, m\}$, *such that there is* $i_0 \in \{1, 2, \cdots, m\}$ *with* $A_{i_0} \in P_{cl}(X)$. *Denote* $Y := \bigcup_{i=1}^{m} A_i$ *and consider* $f : Y \to Y$. *Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* f*;*

(ii) There exists a matrix $S \in \mathcal{M}_{kk}(\mathbb{R})$, *with* $S^n \to 0$, *as* $n \to +\infty$ *such that* $d(f(x_1), f(x_2)) \leq S \cdot d(x_1, x_2)$, *for each* $x_1 \in A_i$, *and* $x_2 \in A_{i+1}$, *for* $i \in \{1, 2, \cdots, m\}$, *where* $A_{m+1} = A_1$.

Then $F_{f^m} \neq \emptyset$.

References

[1] J. Dugundji, A. Granas, *Fixed point theory*, Springer Verlag, Berlin, 2003.

[2] W. A. Kirk, B. Sims (editors), *Handbook of metric fixed point theory*, Kluwer Acad. Publ., Dordrecht, 2001.

[3] W. A. Kirk, P. S. Srinivasan, P. Veeramani, *Fixed poinys for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, **4**(2003), 79-89.

[4] I. A. Rus, *Generalized contractions and applications*, Cluj Univ. Press, 2001.

[5] I. A. Rus, *Cyclic representations and fixed points*, Annals of the Tiberiu Popoviciu Seminar, 2005, to appear.

[6] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory 1950-2000: Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

Babeş-Bolyai University, Faculty of Mathematics
and Computer Science, Str. Kogălniceanu, 1
Cluj-Napoca, Romania
E-mail address: gabip@math.ubbcluj.ro