STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 3, September 2005

EINSTEIN EQUATIONS IN THE GEOMETRY OF SECOND ORDER

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Abstract. In [7], R. Miron and Gh. Atanasiu wrote the Einstein equations of a metric structure G on the tangent bundle of order two, T^2M (previously named "2-osculator bundle" and denoted by Osc^2M), endowed with a nonlinear connection N and a linear connection D such that the 2-tangent structure J be absolutely parallel to D.

In the present paper, the authors determine the Einstein equations by making use of the concept of N-linear connection defined by Gh. Atanasiu, [1], this is, a linear connection which is not neccesarily compatible with J, but only preserves the distributions generated by the nonlinear connection N.

1. The Tangent Bundle T^2M

Let M be a real n- dimensional manifold of class $\mathcal{C}^\infty$, $\left(T^2M, \pi^2, M\right)$ its second order tangent bundle and let $\widetilde{T^2M}$ be the space T^2M without its null section. For a point $u \in T^2M$, let $(x^a, y^{(1)a}, y^{(2)a})$ be its coordinates in a local chart.

Let N be a nonlinear connection, [3, 8-13], and denote its coefficients by $\left(\underset{1}{N}{}^a_{\ b}, \underset{2}{N}{}^a_{\ b}\right)$, a, $b = 1, ..., n$.Then, N determines the direct decomposition

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2M. \tag{1}$$

Received by the editors: 15.09.2005.

2000 *Mathematics Subject Classification.* 53C60, 58 B20, 70 G45.

Key words and phrases. 2-tangent bundle, nonlinear connection, N-linear connection, Riemannian metric, Ricci tensor, Einstein equations.

The adapted basis to (1) is $(\delta_a, \delta_{1a}, \delta_{2a})$ and its dual basis is $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, where

$$\begin{cases} \delta_a = \dfrac{\delta}{\delta x^a} = \dfrac{\partial}{\partial x^a} - \underset{1}{N}{}_a^c \dfrac{\partial}{\partial y^{(1)c}} - \underset{2}{N}{}_a^c \dfrac{\partial}{\partial y^{(2)c}} \\ \delta_{1a} = \dfrac{\delta}{\delta y^{(1)a}} = \dfrac{\partial}{\partial y^{(1)a}} - \underset{1}{N}{}_a^c \dfrac{\partial}{\partial y^{(2)c}} \\ \delta_{2a} = \dfrac{\partial}{\partial y^{(2)a}}, \end{cases} \tag{2}$$

respectively,

$$\begin{cases} \delta y^{(1)a} = dy^{(1)a} + \underset{1}{M}{}_a^c dx^c \\ \delta y^{(2)a} = dy^{(2)a} + \underset{1}{M}{}_a^c dy^{(1)c} + \underset{2}{M}{}_a^c dx^c, \end{cases} \tag{3}$$

where $\underset{1}{M}{}_a^c, \underset{2}{M}{}_a^c$ are the dual coefficients of the nonlinear connection N.

Then, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)a}\delta_a + X^{(1)a}\delta_{1a} + X^{(2)a}\delta_{2a}, \tag{4}$$

with the three right terms (called *d-vector fields*) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_a^{(0)} dx^a + \omega_a^{(1)} \delta y^{(1)a} + \omega_a^{(2)} \delta y^{(2)a}.$$

Similarly, a tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (1) into components ,which will be called *d-tensor fields.*

The $\mathcal{F}\left(T^2M\right)$-linear mapping $J : \mathcal{X}\left(T^2M\right) \to \mathcal{X}\left(T^2M\right)$given by

$$J\left(\delta_a\right) = \delta_{1a}, J\left(\delta_{1a}\right) = \delta_{2a}, J\left(\delta_{2a}\right) = 0 \tag{5}$$

is called the *2-tangent structure on* T^2M,[8-13].

2. N-linear connections. d-tensors of curvature

An *N-linear connection D*, [1], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2. Let us notice that an N-linear connection, in the sense of the definition above, is not necessarily compatible to the

2-tangent structure J (an N-linear connection which is also compatible to J is called, [1], a *JN-linear connection*).

An N-linear connection is locally given by its coefficients

$$D\Gamma(N) = \left(\underset{(00)}{L}{}^a{}_{bc}, \underset{(10)}{L}{}^a{}_{bc}, \underset{(20)}{L}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}^a{}_{bc}, \underset{(12)}{C}{}^a{}_{bc}, \underset{(22)}{C}{}^a{}_{bc} \right), \tag{6}$$

where

$$\begin{cases} D_{\delta_c}\delta_b = \underset{(00)}{L}{}^a{}_{bc}\delta_a,\ D_{\delta_c}\delta_{1b} = \underset{(10)}{L}{}^a{}_{bc}\delta_{1a},\ D_{\delta_c}\delta_{2b} = \underset{(20)}{L}{}^a{}_{bc}\delta_{2a} \\ D_{\delta_{1c}}\delta_b = \underset{(01)}{C}{}^a{}_{bc}\delta_a,\ D_{\delta_{1c}}\delta_{1b} = \underset{(11)}{C}{}^a{}_{bc}\delta_{1a},\ D_{\delta_{1c}}\delta_{2b} = \underset{(21)}{C}{}^a{}_{bc}\delta_{2a} \\ D_{\delta_{2c}}\delta_b = \underset{(02)}{C}{}^a{}_{bc}\delta_a,\ D_{\delta_{2c}}\delta_{1b} = \underset{(12)}{C}{}^a{}_{bc}\delta_{1a},\ D_{\delta_{2c}}\delta_{2b} = \underset{(22)}{C}{}^a{}_{bc}\delta_{2a} \end{cases}. \tag{7}$$

In the particular case when D is J-compatible, we have

$$\begin{aligned} \underset{(00)}{L}{}^a{}_{bc} &= \underset{(10)}{L}{}^a{}_{bc} = \underset{(20)}{L}{}^a{}_{bc} =: L^a{}_{bc}, \\ \underset{(01)}{C}{}^a{}_{bc} &= \underset{(11)}{C}{}^a{}_{bc} = \underset{(21)}{C}{}^a{}_{bc} = \underset{(1)}{C}{}^a{}_{bc}, \\ \underset{(02)}{C}{}^a{}_{bc} &= \underset{(12)}{C}{}^a{}_{bc} = \underset{(22)}{C}{}^a{}_{bc} = \underset{(2)}{C}{}^a{}_{bc}. \end{aligned}$$

For an N-linear connection, let

$$\begin{cases} \underset{0}{D}{}^H_X Y = D_{X^H} Y^H,\ \underset{0}{D}{}^{V_1}_X Y = D_{X^{V_1}} Y^H,\ \underset{0}{D}{}^{V_2}_X Y = D_{X^{V_2}} Y^H \\ \underset{\beta}{D}{}^H_X Y = D_{X^H} Y^{V_\beta},\ \underset{\beta}{D}{}^{V_1}_X Y = D_{X^{V_1}} Y^{V_\beta},\ \underset{\beta}{D}{}^{V_2}_X Y = D_{X^{V_2}} Y^{V_\beta}, \\ \qquad\qquad \beta = 1, 2. \end{cases}$$

$\underset{\alpha}{D}{}^H$, $\underset{\alpha}{D}{}^{V_1}$, $\underset{\alpha}{D}{}^{V_2}$ are called respectively, h_α-, $v_{1\alpha}$- and $v_{2\alpha}$-*covariant derivatives*, $\alpha = 0, 1, 2$. In local coordinates, for a d-tensor field

$$T = T^{a_1 \dots a_r}_{b_1 \dots b_s}\left(x, y^{(1)}, y^{(2)}\right) \delta_{a_1} \otimes \dots \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}.$$

we have

$$\underset{\alpha}{D}{}^H_X T = X^{(0)m}\, T^{a_1 \dots a_r}_{b_1 \dots b_s|_{\alpha m}} \delta_{a_1} \otimes \dots \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s},$$

where

$$T^{a_1...a_r}_{b_1...b_s|\alpha m} = \delta_m T^{a_1...a_r}_{b_1...b_s} + \underset{(\alpha 0)}{L}{}^{a_1}_{hm} T^{ha_2...a_r}_{b_1...b_s} + ... \underset{(\alpha 0)}{L}{}^{a_r}_{hm} T^{a_1...a_{r-1}h}_{b_1...b_s} -$$
$$- \underset{(\alpha 0)}{L}{}^{h}_{b_1 m} T^{a_1...a_r}_{hb_2...b_s} - ... - \underset{(\alpha 0)}{L}{}^{h}_{b_s m} T^{a_1...a_r}_{b_1...b_{s-1}h}.$$

and

$$\underset{\alpha}{D}{}^{V_\beta}_X T = X^{(1)m}\, T^{a_1...a_r}_{b_1...b_s} \overset{(\beta)}{|}_{\alpha m}\ \delta_{a_1} \otimes ... \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes ... \otimes \delta y^{(2)b_s},$$

where

$$T^{a_1...a_r}_{b_1...b_s} \overset{(\beta)}{|}_{\alpha m} = \delta_{\beta m} T^{a_1...a_r}_{b_1...b_s} + \underset{(\alpha\beta)}{C}{}^{a_1}_{hm} T^{ha_2...a_r}_{b_1...b_s} + ... \underset{(\alpha\beta)}{C}{}^{a_r}_{hm} T^{a_1...a_{r-1}h}_{b_1...b_s} -$$
$$- \underset{(\alpha\beta)}{C}{}^{h}_{b_1 m} T^{a_1...a_r}_{hb_2...b_s} - ... - \underset{(\alpha\beta)}{C}{}^{h}_{b_s m} T^{a_1...a_r}_{b_1...b_{s-1}h}.$$

The curvature of the N-linear connection D,

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

is completely determined by its components (which are d-tensors) $R(\delta_{\gamma l}, \delta_{\beta k})\delta_{\alpha j}$. Namely, the 2-forms of curvature of an N- linear connection are, [1],

$$\underset{(\alpha)}{\Omega}{}^a_b = \frac{1}{2}\underset{(0\alpha)}{R}{}_b{}^a{}_{cd} dx^c \wedge dx^d + \underset{(1\alpha)}{P}{}_b{}^a{}_{cd} dx^c \wedge \delta y^{(1)d} + \underset{(2\alpha)}{P}{}_b{}^a{}_{cd} dx^c \wedge \delta y^{(2)d} + \qquad (8)$$
$$\frac{1}{2}\underset{(1\alpha)}{S}{}_b{}^a{}_{cd} \delta y^{(1)c} \wedge \delta y^{(1)d} + \underset{(2\alpha)}{Q}{}_b{}^a{}_{cd} dy^{(1)c} \wedge \delta y^{(2)d} + \frac{1}{2}\underset{(2\alpha)}{S}{}_b{}^a{}_{cd} \delta y^{(2)c} \wedge \delta y^{(2)d},$$

$\alpha = 0,1,2$, where the coefficients $\underset{(0\alpha)}{R}{}_b{}^a{}_{cd}$, $\underset{(\beta\alpha)}{P}{}_b{}^a{}_{cd}$, $\underset{(\beta\alpha)}{Q}{}_b{}^a{}_{cd}$, $\underset{(\beta\alpha)}{S}{}_b{}^a{}_{cd}$ are d-tensors, named the *d-tensors of curvature* of the N-linear connection D. For a JN- linear connection, there holds

$$\underset{(0)}{\Omega}{}^a_b = \underset{(1)}{\Omega}{}^a_b = \underset{(2)}{\Omega}{}^a_b,$$

this is,

$$\underset{(00)}{R}{}_b{}^a{}_{cd} = \underset{(01)}{R}{}_b{}^a{}_{cd} = \underset{(02)}{R}{}_b{}^a{}_{cd} = \underset{(0)}{R}{}_b{}^a{}_{cd};$$
$$\underset{(\beta 0)}{P}{}_b{}^a{}_{cd} = \underset{(\beta 1)}{P}{}_b{}^a{}_{cd} = \underset{(\beta 2)}{P}{}_b{}^a{}_{cd} = \underset{(\beta)}{P}{}_b{}^a{}_{cd} \qquad (9)$$

$$\underset{(20)}{Q}{}_{b\,cd}^{\;a} = \underset{(21)}{Q}{}_{b\,cd}^{\;a} = \underset{(22)}{Q}{}_{b\,cd}^{\;a} = Q_{b\,cd}^{\;a}$$
$$\underset{(\beta 0)}{S}{}_{b\,cd}^{\;a} = \underset{(\beta 1)}{S}{}_{b\,cd}^{\;a} = \underset{(\beta 2)}{S}{}_{b\,cd}^{\;a} = \underset{(\beta)}{S}{}_{b\,cd}^{\;a}, \beta = 1,2.$$

The detailed expressions of the d-tensors of curvature can be found in [1].

3. Metric structures on T^2M

A *Riemannian metric* on T^2M is a tensor field G of type $(0,2)$, which is nondegenerate in each $u \in T^2M$ and is positively defined on T^2M.

In this paper, we shall consider metrics in the form

$$G = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}{}_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}, \tag{10}$$

where $\underset{(\alpha)}{g}{}_{ab} = \underset{(\alpha)}{g}{}_{ab}(x, y^{(1)}, y^{(2)})$; this is, such that the distributions N, N_1 and V_2 generated by the nonlinear connection N be orthogonal with respect to G.

An N-linear connection D is called a *metrical* N-linear connection if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$.

This means

$$\underset{(\alpha)}{g}{}_{ab|\alpha c} = \underset{(\alpha)}{g}{}_{ab} \overset{\beta}{|}_{\alpha c} = 0,\ \alpha = 0,1,2,\ \beta = 1,2.$$

The existence of metrical $N-$linear connections is proved in [2].

4. The Ricci tensor $Ric\,(D)$

Let us notice that, if D is not J- compatible, we could expect that the components of the Ricci tensor look in a more complicated way that the ones in the Miron-Atanasiu theory, [7].

Indeed, if we consider the Ricci tensor $Ric\,(D)$,[14], as the trace of the linear operator

$$V \mapsto R\,(V,X)\,Y,\ \forall V = V^{(0)a}\delta_a + V^{(1)a}\delta_{1a} + V^{(2)a}\delta_{2a} \in \mathcal{X}\left(T^2M\right), \tag{11}$$

then we have:

$$Ric\,(D)\,(X,Y) = trace(V \mapsto R\left(V^{H},X\right)Y + R\left(V^{V_1},X\right)Y + +R\left(V^{V_1},X\right)Y). \quad (12)$$

By a straightforward calculus, we obtain:

Theorem 4.1. *The Ricci tensor $Ric\,(d)$ has the following components:*

$$\begin{aligned}
Ric\,(D)\left(\frac{\delta}{\delta x^b},\frac{\delta}{\delta x^a}\right) &= \underset{(00)}{R}{}_{a\ bc}^{\ c} =: R_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)b}},\frac{\delta}{\delta x^a}\right) &= -\underset{(10)}{P}{}_{a\ cb}^{\ c} =: -\overset{2}{\underset{(10)}{P}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)b}},\frac{\delta}{\delta x^a}\right) &= -\underset{(20)}{P}{}_{a\ cb}^{\ c} =: -\overset{2}{\underset{(20)}{P}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta x^b},\frac{\delta}{\delta y^{(1)a}}\right) &= \underset{(11)}{P}{}_{a\ bc}^{\ c} =: \overset{1}{\underset{(11)}{P}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)b}},\frac{\delta}{\delta y^{(1)a}}\right) &= \underset{(11)}{S}{}_{a\ bc}^{\ c} =: \underset{(1)}{S}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)b}},\frac{\delta}{\delta y^{(1)a}}\right) &= -\underset{(21)}{Q}{}_{a\ cb}^{\ c} =: -\overset{2}{\underset{(21)}{Q}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta x^b},\frac{\delta}{\delta y^{(2)a}}\right) &= \underset{(22)}{P}{}_{a\ bc}^{\ c} =: \overset{1}{\underset{(22)}{P}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)b}},\frac{\delta}{\delta y^{(2)a}}\right) &= \underset{(22)}{Q}{}_{a\ bc}^{\ c} =: \overset{1}{\underset{(22)}{Q}}{}_{ab};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)b}},\frac{\delta}{\delta y^{(2)a}}\right) &= \underset{(22)}{S}{}_{a\ bc}^{\ c} =: \underset{(2)}{S}{}_{ab}.
\end{aligned}$$

The Ricci scalar $Sc\,(D)$ is, thus,

$$Sc\,(D) = g^{ab}R_{ab} + \underset{(1)}{g}{}^{ab}\underset{(1)}{S}{}_{ab} + \underset{(2)}{g}{}^{ab}\underset{(2)}{S}{}_{ab}, \quad (13)$$

where g^{ab}, $\underset{(1)}{g}{}^{ab}$, $\underset{(2)}{g}{}^{ab}$ are the coefficients of the inverse matrix of G.

In the particular case of a JN-linear connection, taking into account (8′), with the notations in [7], we have

$$\underset{(\beta\beta)}{\overset{1}{P}}{}_{ab} = \underset{(\beta)}{\overset{1}{P}}{}_{ab},\ \underset{(\beta 0)}{\overset{2}{P}}{}_{ab} = \underset{(\beta)}{P}{}_{ab},\ \underset{(22)}{\overset{1}{Q}}{}_{ab} = \underset{(21)}{P}{}^{1}_{ab}\,(= Q_{a\,bc}^{\ \ c}), \tag{14}$$
$$\underset{(21)}{\overset{2}{Q}}{}_{ab} = \underset{(21)}{P}{}^{2}_{ab}\,(= Q_{a\ cb}^{c}).$$

5. **Einstein equations**

The Einstein equations associated to the metrical N-linear connection D are

$$Ric\,(D) - \frac{1}{2}Sc\,(D)\,G = \kappa\mathcal{T}, \tag{15}$$

where κ is a constant and $\mathcal{T}$ is the energy-momentum tensor, given by its components

$$\underset{(\alpha\beta)}{\mathcal{T}}{}_{ab} = \mathcal{T}(\delta_{\beta b}, \delta_{\alpha a})$$

Expressing the above relation in the adapted frame (2), we obtain

Theorem 5.1. *The Einstein equations associated to the metrical N- linear connection D are*

$$R_{ab} - \tfrac{1}{2}Sc\,(D)\,g_{ab} = \kappa\,\underset{(00)}{\mathcal{T}}{}_{ab};$$
$$\underset{(\beta\beta)}{\overset{1}{P}}{}_{ab} = \kappa\,\underset{(\beta 0)}{\mathcal{T}}{}_{ab},\ \beta = 1, 2;$$
$$\underset{(\beta 0)}{\overset{2}{P}}{}_{ab} = -\kappa\,\underset{(0\beta)}{\mathcal{T}}{}_{ab},\ \beta = 1, 2;$$
$$\underset{(\beta)}{S}{}_{ab} - \tfrac{1}{2}Sc\,(D)\,\underset{(\beta)}{g}{}_{ab} = \kappa\,\underset{(\beta\beta)}{\mathcal{T}}{}_{ab},\ \alpha = 1, 2;$$
$$\underset{(22)}{\overset{1}{Q}}{}_{ab} = \kappa\,\underset{(21)}{\mathcal{T}}{}_{ab};$$
$$\underset{(21)}{\overset{2}{Q}}{}_{ab} = -\kappa\,\underset{(12)}{\mathcal{T}}{}_{ab}.$$

In the case when D is a JN-linear connection, one obtains the result in [7].

In order to avoid confusions when raising and lowering indices, because of the fact that the components g^{ab}, $\underset{(1)}{g}{}^{ab}$, $\underset{(2)}{g}{}^{ab}$ are different, we will denote in the following by $i, j, ...$ the indices corresponding to the horizontal distribution, by $a, b, ...$ those corresponding to N_1, and by $p, q, ...$ those corresponding to V_2. Thus, if we impose the condition that the divergence of the energy- momentum tensor vanish, in the adapted frame we will obtain

Theorem 5.2. *The law of conservation on T^2M endowed with the metrical N-linear connection D is given by*

$$\left(R^i_j - \frac{1}{2}Sc(D)\,\delta^i_j\right)_{|i} + \underset{(11)}{\overset{1}{P}}{}^a{}_j \overset{(1)}{|}_a - \underset{(10)}{\overset{2}{P}}{}^a{}_j \overset{(1)}{|}_a + \underset{(22)}{\overset{1}{P}}{}^p{}_j \overset{(2)}{|}_p - \underset{(20)}{\overset{2}{P}}{}^p{}_j \overset{(2)}{|}_p = 0;$$

$$\underset{(11)}{\overset{1}{P}}{}^i{}_{b|i} - \underset{(10)}{\overset{2}{P}}{}^i{}_{b|i} + \left(\underset{(1)}{S}{}^a{}_b - \frac{1}{2}Sc(D)\,\delta^a_b\right) \overset{(1)}{|}_a + \underset{(22)}{\overset{1}{Q}}{}^p{}_b \overset{(2)}{|}_p - \underset{(21)}{\overset{2}{Q}}{}^p{}_b \overset{(2)}{|}_p = 0;$$

$$\underset{(22)}{\overset{1}{P}}{}^i{}_{p|i} - \underset{(20)}{\overset{2}{P}}{}^i{}_{p|i} + \underset{(22)}{\overset{1}{Q}}{}^a{}_p \overset{(2)}{|}_a - \underset{(21)}{\overset{2}{Q}}{}^a{}_p \overset{(2)}{|}_a + \left(\underset{(2)}{S}{}^a{}_b - \frac{1}{2}Sc(D)\,\delta^p_q\right) \overset{(2)}{|}_p = 0.$$

In the same way, one can deduce the Maxwell equations associated to the metrical N-linear connection D.

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