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TWO THEOREMS ON KENMOTSU HYPERSURFACES IN A W_3-MANIFOLD

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Abstract. A criterion of the minimality of a Kenmotsu hypersurface in a special Hermitian manifold is established. It is also proved that a Kenmotsu hypersurface in a special Hermitian manifold is minimal if and only if its type number is even.

1. **Introduction**

The theory of almost contact metric structures occupies one of the leading places in modern differential-geometrical researches. It is due to a number of its applications in mathematical physics (for example, in classical mechanics [1] and in theory of geometrical quantization [7]). Furthermore, we mark out the richness of the internal contents of the theory of almost contact metric structures as well as the close connection of this theory with other sections of geometry.

We recall that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi, \xi, \eta, g\}$ on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of the type $(1,1)$ and $g = \langle \cdot, \cdot \rangle$ is the Riemannian metric. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N),$$

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where $\aleph(N)$ is the module of smooth vector fields on N. As an example of an almost contact metric structure we can consider the cosymplectic structure, that is characterized by the following condition:

$$\nabla\eta = 0, \quad \nabla\Phi = 0,$$

where ∇ is the Levi-Civita connection of the metric. It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to a product $M \times R$, where M is a Kählerian manifold [10].

The almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on $N \times R$ [5]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian [5]. On the other hand, we can characterize the Sasakian struture by the following condition:

$$\nabla_X(\Phi)Y = \langle X, Y\rangle\xi - \eta(Y)X, \quad X, Y \in \aleph(N). \tag{1}$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kählerian manifold [5]. As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry.

In 1972 Katsuei Kenmotsu has introduced a new class of almost contact metric structures [8], defined by the condition

$$\nabla_X(\Phi)Y = \langle \Phi X, Y\rangle\xi - \eta(Y)\Phi X, \quad X, Y \in \aleph(N). \tag{2}$$

The Kenmotsu manifolds are normal and integrable, but they are not contact, consequently, they can not be Sasakian. In spite of the fact that the conditions (1) and (2) are similar, the properties of Kenmotsu manifolds are to some extent antipodal to the Sasakian manifolds properties [9]. Note that the new investigation [9] in this field contains a detailed description of Kenmotsu manifolds as well as a collection of examples of such manifolds.

In the present paper, Kenmotsu hypersurfaces in W_3-manifolds are considered. This note is a continuation of research of the authors (for example, the second author studied six-dimensional W_3-manifolds before [3], [4]). We remark that the class of W_3-manifolds is one of the most important classes of almost Hermitian manifolds [6]. However, it has been studied not so detailed as other so-called "small" classes of almost Hermitian manifolds. Some dozens of significant works are devoted to the nearly-Kählerian, almost Kählerian and locally conformal Kählerian manifolds, but much less of articles are written about W_3-manifolds.

2. **Preliminaries**

We consider an almost Hermitian manifold M^{2n}, i.e. a $2n$-dimensional manifold with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold:

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞. We recall that the fundamental (or Kählerian) form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

Let $\big(M^{2n}, \{J, g = \langle \cdot, \cdot \rangle\}\big)$ be an arbitrary almost Hermitian manifold. We fix a point $p \in M^{2n}$. As $T_p(M^{2n})$ we denote the tangent space at the point p, $\{J_p, g_p = \langle \cdot, \cdot \rangle\}$ is the almost Hermitian structure at the point p induced by the structure $\{J, g = \langle \cdot, \cdot \rangle\}$. The frames adapted to the structure (or the A-frames) look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\widehat{1}}, \ldots, \varepsilon_{\widehat{n}}),$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\widehat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$ [2]. Here the index a ranges from 1 to n, and we state $\widehat{a} = a + n$.

The matrix of the operator of the almost complex structure written in an A-frame looks as follows:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$. By direct computing it is easy to obtain that the matrices of the metric g and of the fundamental form F in an A-frame look as follows, respectively:

$$(g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad (F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

An almost Hermitian manifold is called special Hermitian, if

$$\delta F = 0, \quad \nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0, \quad X, Y, Z \in \aleph(M^{2n}),$$

where δ is the codifferentiation operator. The first group of the Cartan structural equations of a special Hermitian manifold written in an A-frame looks as follows:

$$d\omega^a = \omega_b^a \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b,$$

$$d\omega_a = -\omega_a^b \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b,$$

and moreover,

$$B^{ab}{}_b = 0, \quad B_{ab}{}^b = 0, \tag{3}$$

where $\{B^{ab}{}_c\}$ and $\{B_{ab}{}^c\}$ are components of the Kirichenko tensors of M^{2n} [2], $a, b, c = 1, \ldots, n$.

3. The main results

Theorem 3.1. Let N be a Kenmotsu hypersurface in a special Hermitian manifold M^{2n}, and let σ be the second fundamental form of the immersion of N into M^{2n}. Then N is a minimal submanifold of M^{2n} if and only if $\sigma(\xi, \xi) = 0$.

Proof. Let us use the Cartan structural equations of an almost contact metric structure on a hypersurface in a Hermitian manifold [4]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + (\sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta)\omega^\beta \wedge \omega +$$

$$+(-\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n + i\sigma^{\alpha\beta})\omega_\beta \wedge \omega,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + (\sqrt{2}B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha)\omega_\beta \wedge \omega +$$

$$+(-\frac{1}{\sqrt{2}}B_{\alpha\beta}{}^n - i\sigma_{\alpha\beta})\omega^\beta \wedge \omega,$$

$$d\omega = (\sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta)\omega^\beta \wedge \omega_\alpha + (B_{n\beta}{}^n + i\sigma_{n\beta})\omega \wedge \omega^\beta +$$

$$+(B^{n\beta}{}_n - i\sigma^\beta_n)\omega \wedge \omega_\beta.$$

Here and further the indices α, β, γ range from 1 to $n-1$.

Taking into account that the Cartan structural equations of a Kenmotsu structure look as follows [9]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + \omega \wedge \omega^\alpha,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + \omega \wedge \omega_\alpha,$$

$$d\omega = 0,$$

we get the conditions, whose simultaneous fulfillment is a criterion for the structure on N to be Kenmotsu:

$$1)\ B^{\alpha\beta}{}_\gamma = 0;\quad 2)\ \sqrt{2}B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta = -\delta^\alpha_\beta;\quad 3)\ -\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_n + i\sigma^{\alpha\beta} = 0;$$

$$4)\ \sqrt{2}B^{n\alpha}{}_\beta - \sqrt{2}B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta = 0;\quad 5)\ B^{n\beta}{}_n - i\sigma^\beta_n = 0 \tag{4}$$

and the formulae obtained by the complex conjugation (no need to write them explicitly). From $(4)_3$ we have:

$$\sigma^{\alpha\beta} = -\frac{i}{\sqrt{2}}B^{\alpha\beta}{}_n.$$

Since

$$0 = \sigma^{[\alpha\beta]} = -\frac{i}{\sqrt{2}}B^{[\alpha\beta]}{}_n = -\frac{i}{2\sqrt{2}}(B^{\alpha\beta}{}_n - B^{\beta\alpha}{}_n) = -\frac{i}{\sqrt{2}}B^{\alpha\beta}{}_n,$$

we get $B^{\alpha\beta}{}_n = 0$, and that is why

$$\sigma^{\alpha\beta} = 0.$$

Similarly, from $(4)_5$ we obtain

$$\sigma_n^{\beta} = 0.$$

Therefore we can rewrite the conditions (4) as follows:

$$1)\ B^{\alpha\beta}{}_\gamma = 0;\quad 2)\ \sigma^{\alpha\beta} = 0;\quad 3)\ \sigma_n^{\beta} = 0;\quad 4)\ \sigma_\beta^\alpha = i\sqrt{2}B^{\alpha n}{}_\beta + i\delta_\beta^\alpha \tag{5}$$

and the formulae obtained by the complex conjugation.

Now, let us use a criterion of the minimality of an arbitrary hypersurface [11]:

$$g^{ps}\sigma_{ps} = 0, \quad p,s = 1,2,\dots,2n-1.$$

Knowing how the matrix of the contravariant metric tensor on N looks [3]:

$$(g^{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \dots\\ 0\end{matrix} & I_{n-1} \\ \hline 0\ \dots\ 0 & 1 & 0\ \dots\ 0 \\ \hline I_{n-1} & \begin{matrix}0\\ \dots\\ 0\end{matrix} & 0 \end{array}\right),$$

we obtain:

$$g^{ps}\sigma_{ps} = g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\widehat{\alpha}\widehat{\beta}}\sigma_{\widehat{\alpha}\widehat{\beta}} + g^{\widehat{\alpha}\beta}\sigma_{\widehat{\alpha}\beta} + g^{\alpha\widehat{\beta}}\sigma_{\alpha\widehat{\beta}} + g^{nn}\sigma_{nn} =$$

$$= g^{\widehat{\alpha}\beta}\sigma_{\widehat{\alpha}\beta} + g^{\alpha\widehat{\beta}}\sigma_{\alpha\widehat{\beta}} + g^{nn}\sigma_{nn}.$$

By force of (3) and (5) we have

$$g^{ps}\sigma_{ps} = i\sqrt{2}B^{\alpha n}{}_\alpha + i(n-1) - i\sqrt{2}B_{\alpha n}{}^\alpha - i(n-1) + \sigma_{nn} = \sigma_{nn}.$$

That is why $g^{ps}\sigma_{ps} = 0 \Leftrightarrow \sigma_{nn} = 0$. The last equality means that

$$\sigma(\xi,\xi) = 0. \tag{6}$$

So, a Kenmotsu hypersurface in a W_3-manifold is minimal precisely when (6) holds, Q.E.D.

Since the class of special Hermitian manifolds contains all Kählerian manifolds [6], by force of THEOREM A we come to the following result.

Corollary 3.1. A Kenmotsu hypersurface in a Kählerian manifold is minimal if and only if

$$\sigma(\xi, \xi) = 0.$$

Now, let N be a totally umbilical Kenmotsu hypersurface in a W_3-manifold M^{2n}. Then $\sigma = \lambda g$, λ — *const*, therefore the matrix of the second fundamental form looks as follows:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} \begin{matrix} \\ 0 \\ \\ \end{matrix} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & \begin{matrix} \\ \lambda I_{n-1} \\ \\ \end{matrix} \\ \hline 0 \ \dots \ 0 & \lambda & 0 \ \dots \ 0 \\ \hline \begin{matrix} \\ \lambda I_{n-1} \\ \\ \end{matrix} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & \begin{matrix} \\ 0 \\ \\ \end{matrix} \end{array}\right),$$

As it has been proved, the hypersurface will be minimal if and only if $\lambda = 0$. Evidently, the matrix (σ_{ps}) vanishes in this case, therefore we conclude that N will be a totally geodesic hypersurface in M^{2n}. That is why we have such an additional result.

Corollary 3.2. A totally umbilical Kenmotsu hypersurface in a W_3-manifold is minimal if and only if it is totally geodesic.

As it is well-known (see, for example, [12] or [13]), when we give a Riemannian manifold and its submanifold, the rank of the determined second fundamental form is called the type number. Now, we can state the second main result of this work:

Theorem 3.2. A Kenmotsu hypersurface in a W_3-manifold is minimal if and only if its type number is even.

Proof. Considering the matrix of the second fundamental form of a Kenmotsu hypersurface in a special Hermitian manifold, it is easy to see that this hypersurface is minimal precisely when the following condition holds:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix}0\\ \dots\\ 0\end{matrix} & \sigma_{\alpha\widehat{\beta}} \\ \hline 0\ \dots\ 0 & 0 & 0\ \dots\ 0 \\ \hline \sigma_{\widehat{\alpha}\beta} & \begin{matrix}0\\ \dots\\ 0\end{matrix} & 0 \end{array}\right),$$

Taking into account that [14] $\sigma_{\widehat{\alpha}\beta} = \overline{\sigma_{\alpha\widehat{\beta}}}$, we have:

$$rank(\sigma_{ps}) = 2rank(\sigma_{\alpha\widehat{\beta}}).$$

On the other hand, if the Kenmotsu hypersurface is not minimal, then

$$rank(\sigma_{ps}) = 2rank(\sigma_{\alpha\widehat{\beta}}) + 1.$$

Thus, a Kenmotsu hypersurface in a W_3-manifold is minimal precisely when its type number is even, Q.E.D.

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