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THE UNITARY TOTIENT MINIMUM AND MAXIMUM FUNCTIONS

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Abstract. The unitary totient function has been introduced by E. Cohen [1]. The Euler minimum function has been first studied by P. Moree and H. Roskam [2], and independently by the author [4], who introduced more general concepts (and duals). A particular case is obtained for the unitary totient. Basic properties for this minimum, as well as maximum functions are pointed out. These include inequalities, divisibility properties, and values taken at special arguments. The necessary exponential diophantine equations are treated by elementary arguments.

1. **Introduction**

A divisor d of n is called **unitary** if $\left(d, \frac{n}{d}\right) = 1$. Let $(k, n)_*$ denote the greatest divisor of k which is a unitary divisor of n. The arithmetical functions associated with unitary divisors have been introduced by E. Cohen [1]. The multiplicative function

$$\mu^*(n) = (-1)^{\omega(n)},$$

where $\omega(n)$ denotes the number of distinct prime factors of n, is the unitary analogue of the Möbius function $\mu(n)$ and we have

$$\sum_{d||n} \mu^*(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

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where $d||n$ denotes that d is a unitary divisor of n. Let $\varphi^*(n)$ denote the unitary analogue of the Euler totient function, that is $\varphi^*(n)$ represents the number of positive integers $k \leq n$ with $(k,n)_* = 1$. Then, it is easy to see that

$$\varphi^*(n) = \sum_{d|n} d\mu^*\left(\frac{n}{d}\right),$$

so $\varphi^*(n)$ is multiplicative, being the unitary convolution of two multiplicative functions (see [1]), and for $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$ (prime factorization of n) we have

$$\varphi^*(n) = (p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1) \tag{1}$$

Put $\varphi^*(1) = 1$.

Let $A \subset \mathbb{N}^* = \{1, 2, \dots\}$ be a given set, and $f, g : \mathbb{N}^* \to \mathbb{N}^*$ two given functions. In [4] and [5] we have introduced the functions $F_f^A(n)$, $G_g^A(n)$ by (if these are well-defined)

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\}, \tag{2}$$

and its "dual" by

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\} \tag{3}$$

For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(n)$, one obtains the "Euler minimum" and "Euler maximum" functions, given by

$$E(n) = \min\{k \in \mathbb{N}^* : \ n|\varphi(k)\}, \tag{4}$$

$$E_*(n) = \max\{k \in \mathbb{N}^* : \ \varphi(k)|n\} \tag{5}$$

For properties of $E(n)$ given by (4) see [2] and [6], while function (5) appears for the first time in [4] and [6].

The author has considered also other particular cases of (2) and (3) for $f(k) = g(k) = \sigma(k)$ (sum of divisors of k), $f(k) = d(k)$ (number of divisors of k) [7], $f(k) = g(k) = T(k)$ (product of divisors of k) [9], $f(k) = g(k) = S(k)$ (Smarandache function) [8], $f(k) = g(k) = Z(k)$ (pseudo-Smarandache function) [11], $f(k) = \varphi_e(n)$ (exponential totient function) [10]. It is interesting to note that, for $g(k) = d(k)$ or $g(k) = \varphi_e(n)$ the analogues functions to (5) are not well-defined.

The aim of this paper is the introduction and the initial study of the particular cases $f(k) = g(k) = \varphi^*(k)$, the unitary totient function. In analogy with (4) and (5) define

$$E^*(n) = \min\{k \geq 1 : \ n|\varphi^*(k)\}, \tag{6}$$

$$E^*_*(n) = \max\{k \geq 1 : \ \varphi^*(k)|n\} \tag{7}$$

First note that the functions $E^*(n)$ and $E^*_*(n)$ are well-defined. Indeed, by Dirichlet's theorem on arithmetic progressions, for each $n \geq 1$ there exists $a \geq 1$ so that $k = an + 1$ is a prime (see e.g. [3]). Then, since by (1) $\varphi^*(k) = k - 1 = an$, which is a multiple of n, (6) is well-defined. On the other hand, remark that

$$\varphi(n) \leq \varphi^*(n), \tag{8}$$

with equality only for $n = 1$ and $n =$ squarefree (i.e. product of distinct primes). Since φ and φ^* are multiplicative, (8) follows from

$$\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} \leq p^\alpha - 1 = \varphi^*(p^\alpha)$$

(p prime, $\alpha \geq 1$), where for $\alpha = 1$ there is equality.

Now, since $\varphi(k) > \sqrt{k}$ for $k > 6$ (see e.g. [3]) and from $\varphi^*(k)|n$ it follows $\varphi^*(k) \leq n$, so $\sqrt{k} < n$, implying $k < n^2$. Thus $E^*_*(n) \leq \max\{6, n^2\} < +\infty$, so this function is well-defined, too.

2. **Main results**

Lemma 1. *For all $n \geq 2$ one has*

$$P(n) - 1 \leq \varphi^*(n) \leq n - 1, \tag{9}$$

where $P(n)$ denotes the greatest prime factor of n.

Proof. The left side inequality follows by $(p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1) \geq p_r^{\alpha_r} - 1 \geq p_r - 1$, where $p_1 < p_2 < \dots < p_r$ are the distinct prime factors of n. Then by (1), $\varphi^*(n) \geq p_r - 1 = P(n) - 1$.

For the right side of (9), apply the obvious relation $(1 + y_1) \dots (1 + y_r) \geq 1 + y_1 \dots y_r$ ($y_i > 0$ for $i = 1, 2, \dots, r$) to $y_i = p_i^{\alpha_i} - 1 > 0$. Then we get $p_1^{\alpha_1} \dots p_r^{\alpha_r} \geq$

$1 + (p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1)$, so by (1), the required result follows. Since here there is equality only for $r = 1$, the equality sign in right-side of (9) is attained only for $n =$ prime power. Clearly, for the left side of (9) there is equality for $n =$ prime.

Lemma 2. *Let $r = \omega(n)$ be the number of distinct prime factors of n. Then*

$$\varphi^*(n) \leq (n^{\frac{1}{r}} - 1)^r \text{ for all } n \geq 2 \tag{10}$$

Proof. Apply the Huyggens inequality

$$\sqrt[r]{(1 + y_1) \dots (1 + y_r)} \geq 1 + \sqrt[r]{y_1 \dots y_r} \quad (y_i > 0)$$

to $y_i = p_i^{\alpha_i} - 1$. Then by (1), inequality (10) follows.

Theorem 1.

$$\varphi^*(E_*^*(n))|n|\varphi^*(E^*(n)) \text{ for all } n \geq 1. \tag{11}$$

Particularly,

$$\varphi^*(E_*^*(n)) \leq n, \tag{12}$$

$$\varphi^*(E^*(n)) \geq n. \tag{13}$$

Proof. Let $E^*(n) = k_0$. By Definition (6), $n|\varphi^*(k_0)$. This gives the right side of (11). If $E_*^*(n) = k_1$, then by (7), $\varphi^*(k_1)|n$, so the left side of (11) follows. Relations (12) and (13) are direct consequences of (11).

Corollary 1.

$$E^*(n) \geq (n^{\frac{1}{r}} + 1)^r \geq n + 1, \text{ for } n \geq 2, \tag{14}$$

where $r = \omega(E^*(n))$.

Proof. By (10), $\varphi^*(E^*(n)) \leq (E^*(n)^{\frac{1}{r}} - 1)^r$, so by (13) we get $n \leq (E^*(n)^{\frac{1}{r}} - 1)^r$. This gives the first relation of (14). The second one is a trivial consequence of $(a + b)^r \geq a^r + b^r$ $(a, b > 0,\ r \geq 1)$, which follows e.g. by the binomial theorem.

Corollary 2.

$$P(E_*^*(n)) \leq n + 1, \quad n \geq 2, \tag{15}$$

where $P(m)$ denotes the greatest prime factor of m.

Proof. This is similar to the proof of (14), by applying (12) and the left side of (9).

Remark 1. The weaker inequality of (14), i.e. $E^*(n) \geq n+1$ for $n \geq 2$ follows also by (13) and the right side of (9). This inequality becomes an equality for many values of n, e.g. for $n = 2, 3, 4, 6, 7, 8, 10, 12, 15, 16, 18, 22, \ldots$. Particularly, we prove:

Theorem 2. *If $p \geq 3$ is a prime, then*

$$E^*(p-1) = p \tag{16}$$

Proof. Since $(p-1)|\varphi^*(p)$ (because of $\varphi^*(p) = p-1$), by definition (6) it follows $E^*(p-1) \leq p$. On the other hand, applying $E^*(n) \geq n+1$ for $n = p-1 \geq 2$ one gets $E^*(p-1) \geq p$, so (16) is proved.

Clearly, since $\varphi^*(p)|(p-1)$, too, by (7) and (15) we get

Theorem 3. *For all primes p,*

$$E_*^*(p-1) \geq p, \tag{17}$$

and

$$P(E_*^*(p-1)) \leq p \text{ for } p \geq 3 \tag{18}$$

Remark 2. The exact calculation of $E_*^*(p-1)$ seems difficult. However, the determination of $E_*^*(p)$ is given by the following

Theorem 4.

$$E_*^*(p) = \begin{cases} 6, & \text{if} \quad p = 2, \\ 2, & \text{if} \quad p \geq 3 \text{ is not a Mersenne prime}, \\ 2^n, & \text{if} \quad p = 2^n - 1 \text{ is a Mersenne prime} \end{cases} \tag{19}$$

First we prove the following auxiliary result:

Lemma 3. *Let p be a prime. Then the equation*

$$\varphi^*(x) = p$$

is solvable if and only if $p = 2$ or p is a Mersenne prime (with a single solution).

Proof. If x is composite, $x = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, with $\omega(x) = r \geq 2$, then $\varphi^*(x) = (p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1)$ is always composite, so $\neq p$. If $r = 1$, i.e. $x = q^\alpha$, then $\varphi^*(x) = q^\alpha - 1 = p$ iff $q^\alpha = p + 1$. Now, if $p \geq 3$, then $p + 1$ is even, so we must have $q = 2$, i.e. $p = 2^\alpha - 1 =$ Mersenne prime (see [3]). If $p = 2$, we get $q = 3$, $\alpha = 1$ so x is not composite.

If $x = q$ is a prime, then $\varphi^*(x) = q - 1 = p \Leftrightarrow q = p + 1$, and this is solvable only if $p = 2$, since for $p \geq 3$, $p + 1$ being even, cannot be a prime.

Now, for the proof of (19), let $\varphi^*(k)|p$. Then $\varphi^*(k) = 1$ (i.e. $k = 1$ or 2), or $\varphi^*(k) = p$. Since $p \geq 2$, always, the result follows from Lemma 3, by taking into account the form of the solution, when p is a Mersenne prime.

We now prove:

Lemma 4. *Let $k \geq 1$ be an integer. Then the equation*

$$\varphi^*(x) = 2^k$$

is always solvable, and its solutions are of the form $x = F$, or $x = 2F$, where $F = 9$; a Fermat prime; or the product of distinct Fermat primes.

Proof. Let $x = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, when $\varphi^*(x) = (p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1) = 2^k$ $\Leftrightarrow$ $p_1^{\alpha_1} - 1 = 2^{a_1}, \dots,$ $p_r^{\alpha_r} - 1 = 2^{a_r}$, with $a_1 + \dots + a_r = k$. Thus $p_1^{\alpha_1} = 2^{a_1} + 1, \dots, p_r^{\alpha_r} = 2^{a_r} + 1$, so each p_i $(i = 1, 2, \dots, r)$ is odd, so x must be odd. Since we can have also the case $2^1 - 1 = 2^0$, x could be also of the form $x = 2F$, where F is odd. Therefore we must study an equation of type

$$p^\alpha = 2^a + 1 \quad (a \geq 1) \tag{20}$$

1) If $\alpha = 2m$ is even, then $(p^m - 1)(p^m + 1) = 2^a$ gives $p^m - 1 = 2^u$, $p^m + 1 = 2^v$ $(u + v = a)$, so $2^v - 2^u = 2$, i.e. $2^v = 2(1 + 2^{u-1})$, which is possible only if $u = 1$,

$v = 2$. Then $p^m - 1 = 2$, $p^m + 1 = 4$, giving $p = 3$, $m = 1$, $\alpha = 2$; so $a = 3$ and $x = p^\alpha = 9$.

2) If $\alpha = 2m + 1$ $(m \geq 0)$, for $m = 0$ we get $\alpha = 1$, so $p = 2^a + 1$ is a prime, so it is a Fermat prime (see [3]). Let $m \geq 1$. Then since $p^{2m+1} - 1 = (p-1)(p^{2m} + p^{2m-1} + \cdots + p + 1)$ and p is odd, the second term contains a number of $2m + 1$ odd terms, so it is odd. Thus (20) is impossible.

This finishes the proof of Lemma 4.

Theorem 5. *$E_*^*(2^t) = 2m$, where m is the greatest of the products $(2^{a_1} + 1) \dots (2^{a_r} + 1)$ of Fermat primes, where $a_1 + \cdots + a_r \leq t$.*

Proof. Let $\varphi^*(k)|2^t$. Then $\varphi^*(k) = 2^a$, where $0 \leq a \leq t$. By Lemma 3, the greatest such k is $k = 2m$, where $m = (2^{a_1}+1) \dots (2^{a_r}+1)$, with $a_1 + \cdots + a_r = a \leq t$ and r is maximal (i.e. m is maximal if $a_1 + \cdots + a_r \leq t$).

Example. $E_*^*(8) = 30$.

Indeed, $8 = 2^3$, $a_1 + \cdots + a_r \leq 3 \Leftrightarrow r = 2$, since $2^1 + 1 = 3$, $2^2 + 1 = 5$ are Fermat primes and $1 + 2 = 3$. So $2m = 2 \cdot 3 \cdot 5 = 30$.

Lemma 5. *Let p be a prime. Then*

$$\varphi^*(x) = p^2$$

is solvable iff $p = 2$. The solutions are $x = 5, 10$.

Proof. 1) Let $x = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be odd. Then $(p_1^{\alpha_1} - 1) \dots (p_r^{\alpha_r} - 1) = p^2$ iff

a) $p_1^{\alpha_1} - 1 = p^2$;

b) $p_1^{\alpha_1} - 1 = 1$, $p_2^{\alpha_2} - 1 = p^2$;

c) $p_1^{\alpha_1} - 1 = 1$, $p_2^{\alpha_2} - 1 = 1$, $p_3^{\alpha_3} - 1 = p^2$;

d) $p_1^{\alpha_1} - 1$, $p_2^{\alpha_2 - 1} = p$, $p_3^{\alpha_3} - 1 = p$.

Remark that cases b), c), d) are impossible, since then $p_1 = 2$ always, and this contradicts $x =$ odd. In case a), since p_1 is odd we must have $p =$ even, so $p = 2$. But in this case, $p_1 = 5$, $\alpha_1 = 1$, so $x = 5$.

2) If x is even, then $p_1 = 2$. In case a) we can write $2^{\alpha_1} = p^2 + 1$. For $p = 2, 3, 5$ this is impossible. If $p > 5$, then it is known (see e.g. [3]) that p must have

the forms $p = 6M \pm 1$, so $p^2 = 36k^2 \pm 12k + 1 = 12k(3k \pm 1) + 1 = 24M + 1$, so $p^2 + 1 = 2(12M + 1) \neq 2^{\alpha_1}$.

In case b) $\alpha_1 = 1$, $p_2^{\alpha_2} = 5$ ($p_2 = 5$, $\alpha_2 = 1$) is possible, implying $x = 2{\cdot}5 = 10$.

Cases c), d) cannot hold, since then e.g. in case c) $\alpha_1 = 1$, $\alpha_2 = 1$, $p_2 = p_3$ and this is a contradiction to $p_3 > p_2$. Similarly, in case d).

Theorem 6.

$$E_*^*(p^2) = \begin{cases} 10, & \textit{if} \quad p = 2 \\ 2, & \textit{if} \quad p \geq 3 \textit{ is not a Mersenne prime} \\ 2^k, & \textit{if} \quad p = 2^k - 1 \textit{ is a Mersenne prime} \end{cases}$$

Proof. $\varphi^*(k)|p^2 \quad \Leftrightarrow \quad \varphi^*(k) \in \{1, p, p^2\}$. Now, apply Lemmas 3, 5, and definition (7).

Lemma 6. *Let p be a prime, $k > 1$ an integer. Then the equation*

$$\varphi^*(x) = p^k$$

is solvable only for $p = 2$.

Proof. First we prove an auxiliary result:

Lemma 6'. *If $k > 1$ and p is a prime, $\alpha \geq 1$, then the equation*

$$p^k = 2^\alpha - 1 \tag{21}$$

is not solvable.

Proof. First remark that p must be odd. If $k = 2m + 1$ ($m \geq 1$) is odd, then $p^{2m+1} + 1 = (p + 1)(p^{2m} - \cdots + p + 1)$, where the second term contains an odd number of odd terms, and the signs + or −, so it is odd. Thus (21) is impossible. If $k = 2m$ is even, and $p > 5$, then by $p = 6s \pm 1$ (as in the proof of case 2) of Lemma 5) $p^2 = 24M + 1$, so $p^{2m} = \mathcal{M}24 + 1$, $p^{2m} + 1 = 2(\mathcal{M}12 + 1) \neq 2^\alpha$.

For $p = 3, 5$ we must consider separately equation (21) in case $k = 2m$. So $3^{2m} = 9^m = (8 + 1)^m = M8 + 1$, so $M8 + 2 = 2(M4 + 1) \neq 2^\alpha$. Similarly, $5^{2m} = 25^m = (2n + 1)^m = M24 + 1$, i.e. $5^{2m} + 1 = M24 + 2 = 2(M12 + 1) \neq 2^\alpha$. This finishes the proof of Lemma 6'.

The proof of Lemma 6 is similar to that of Lemma 5.

When x is odd, then $p_1^{\alpha_1} - 1 = p^k$ in case a) so $p = 2$, so by (20). This is possible only when $p_1 = 3$, so $k = 3$.

The other cases, when $p_1^{\alpha_1} - 1 = 1$, etc. are impossible, since $p_1 = 2$, contradiction to $x =$ odd. Similarly the case $p_1^{\alpha_1} - 1 = p, \dots, p_r^{\alpha_r} - 1 = p$, $k = 2$, since then $p_1 = \dots = p_r$, impossible.

When x is even, i.e. $p_1 = 2$, in case a) we get $2^{\alpha_1} - 1 = p^k$, and by (21) this is not solvable.

Theorem 7.

$$E_*^*(p^k) = \begin{cases} 2m, & \text{if} \quad p = 2, \ \text{where } m \text{ is given by Theorem 5}, \\ 2, & \text{if} \quad p \geq 5 \text{ is not a Mersenne prime}, \\ 2^k, & \text{if} \quad p = 2^k - 1 \text{ is a Mersenne prime}. \end{cases}$$

Proof. This is similar to the proof of Theorem 6 (case $k = 2$), but remarking that for $p = 2$ we must use Theorem 5.

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