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GENERATION OF NON-UNIFORM LOW-DISCREPANCY SEQUENCES IN QUASI-MONTE CARLO INTEGRATION

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Abstract. We propose an inversion type method that can be used in Quasi-Monte Carlo integration to generate low-discrepancy sequences with an arbitrary distribution function G. The method is based on the approximation of the inverse of the distribution function by linear Lagrange interpolation or cubic Hermite interpolation. We also give bounds for the G-discrepancy of the generated sequences.

1. Discrepancy and error bounds

In quasi-Monte Carlo integration one approximates $\int_{[0,1]^s} f(x)dx$ by sums of the form $\frac{1}{N}\sum_{k=1}^{N} f(x_k)$, where $f : [0,1]^s \to \mathbb{R}$ and $(x_1,\ldots,x_N)$ is a sequence of deterministic points, with $x_k = (x_k^{(1)},\ldots,x_k^{(s)}) \in [0,1]^s$, $k = 1,\ldots,N$. A well-known measure of the distribution properties of the sequences used in quasi-Monte Carlo integration is the discrepancy.

Definition 1 (discrepancy). *Let $P = (x_1,\ldots,x_N)$ be a sequence of points in $[0,1]^s$. The discrepancy of sequence P is defined as*

$$D_N(P) = \sup_{J\subseteq[0,1]^s} \left|\frac{1}{N}A_N(J,P) - \lambda(J)\right|,$$

where A_N counts the number of elements of sequence P, falling into the interval J, i.e.,

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$$A_N(J,P)=\sum_{k=1}^{N} 1_J(x_k).$$

1_J is the characteristic function of J and λ is the s-dimensional Lebesgue measure. The sequence P is called uniformly distributed if $D_N(P)\to 0$ when $N\to\infty$.

For $s=1$, we may arrange the points $x_1,\ldots,x_N$ of a given point set in nondecreasing order. The following result is due to Niederreiter [10].

Theorem 2. *If $x_0 := 0 \le x_1 \le x_2 \le \ldots \le x_N \le 1 =: x_{N+1}$, then we have the following expression for the discrepancy of sequence $(x_1,\ldots,x_N)$*

$$D_N(x_1,\ldots,x_N)=\frac{1}{N}+\max_{1\le n\le N}\left(\frac{n}{N}-x_n\right)-\min_{1\le n\le N}\left(\frac{n}{N}-x_n\right)=\max_{\substack{0\le i\le N\\ 1\le j\le N+1}}\left|\frac{1}{N}+r_i-r_j\right| \tag{1}$$

where $r_n=\frac{n}{N}-x_n$ for $0\le n\le N+1$.

The monograph by Niederreiter [10] provides a comprehensive overview on discrepancy, low-discrepancy sequences and their properties. Halton [5], Faure [3], [4], Niederreiter [10] and others constructed famous low-discrepancy sequences.

A similar concept of discrepancy can be defined for non-uniformly distributed sequences.

Definition 3 (non-uniform discrepancy). *Consider an s-dimensional distribution on $[0,1]^s$, with distribution function G. Let λ_G be the probability measure corresponding to G. Let $P=(x_1,\ldots,x_N)$ be a sequence of points in $[0,1]^s$. The G-discrepancy of sequence $P=(x_1,\ldots,x_N)$ is defined as*

$$D_{N,G}(P)=\sup_{J\subseteq[0,1]^s}\left|\frac{1}{N}A_N(J,P)-\lambda_G(J)\right|.$$

The sequence P is called G-distributed, if $D_{N,G}(P)\to 0$ when $N\to\infty$.

If f is a function with finite variation in the sense of Hardy and Krause, $V_{HK}(f)<+\infty$ (see eg. Owen [12]), then an upper bound for the error of the approximation in quasi-Monte Carlo integration is given by the non-uniform Koksma-Hlawka inequality (see Chelson [1]).

Theorem 4 (non-uniform Koksma-Hlawka inequality). *Let* $f : [0,1]^s \to \mathbb{R}$ *be of bounded variation in the sense of Hardy and Krause. Moreover, let* G *be a distribution function with continuous density on* $[0,1]^s$ *and* $(x_1, \ldots, x_N)$ *a sequence on* $[0,1]^s$*. Then, for any* $N > 0$

$$\left| \frac{1}{N} \sum_{k=1}^{N} f(x_k) - \int_{[0,1]^s} f(x) dG(x) \right| \leq V_{HK}(f) D_{N,G}(x_1, \ldots, x_N),$$

where $V_{HK}(f)$ *is the variation of* f *in the sense of Hardy and Krause.*

2. Inversion method

The inversion method produces a random variable with desired distribution function by making use of the inverse of the distribution function.

Consider a distribution on $[0,1]$ with continuous density function g and distribution function $G(x) = \int_0^x g(t)dt$. Assume that there exists the inverse function G^{-1}. The inversion method is based on the following principle.

Theorem 5. *Let* U *be a random variable uniformly distributed on* $[0,1]$*. Then the distribution function of the random variable* $G^{-1}(U)$ *is* G*.*

Proof. We denote by $F_{G^{-1}(U)}$ the distribution function of $G^{-1}(U)$. We have

$$F_{G^{-1}(U)}(x) = P(G^{-1}(U) < x) = P(U < G(x)) = G(x).$$

Thus, the distribution function of the random variable $G^{-1}(U)$ is G. □

Such a transformation preserves the discrepancy in one dimension, as showed in the following theorem (see Okten [11]).

Theorem 6. *Let* $P = (x_1, \ldots, x_N)$ *be a sequence in* $[0,1]$ *and* G *a distribution function on* $[0,1]$*.*

Construct the sequence $(y_1, \ldots, y_N) = (G^{-1}(x_1), \ldots, G^{-1}(x_N))$*. Then the G-discrepancy of the constructed sequence is given by*

$$D_{N,G}(y_1, \ldots, y_N) = D_N(x_1, \ldots, x_N).$$

In other words,

$$D_{N,G}(G^{-1}(P)) = D_N(P).$$

As a consequence, for generating low-discrepancy sequences with an arbitrary distribution function G, we can transform uniformly distributed low-discrepancy sequences using the inverse function G^{-1}. In most cases, however, the inverse G^{-1} cannot be given analytically. In such cases, we may use the inversion technique with an approximation of the inverse function G^{-1}.

3. Existing inversion type methods

The inversion type transformations presented in this paper are designed for the one-dimensional case. They all propose different modalities of approximating the inverse G^{-1}.

3.1. Hlawka-Mück method. Hlawka [7] defines a transformation and bounds the G-discrepancy of the transformed sequence as follows.

Theorem 7. *Consider a continuous type distribution on* $[0,1]$*, with density* g *and distribution function* G*. Assume that the distribution function* G *is invertible and* $M = \sup_{x\in[0,1]} g(x) < \infty$*. Furthermore, let* $(x_1, x_2, \ldots, x_N)$ *be a sequence in* $[0,1]$*. Generate the point set* $(y_1, y_2, \ldots, y_N)$ *with*

$$y_k = \frac{1}{N}\sum_{r=1}^{N}\left[1 + x_k - G(x_r)\right] = \frac{1}{N}\sum_{r=1}^{N} 1_{[0,x_k]}(G(x_r)), \tag{2}$$

where $[a]$ *denotes the integer part of* a*. Then the generated sequence has a* G*-discrepancy of*

$$D_{N,G}(y_1, \ldots, y_N) \leq (2 + 6M) D_N(x_1, \ldots, x_N). \tag{3}$$

This method is known in the literature as the Hlawka-Müch method and it is a generalization of an earlier version proposed by Hlawka and Mück in [8], [9]. The main disadvantage of the Hlawka-Müch method is that all the points of the sequence

$(y_1, y_2, \ldots, y_N)$ are of the form $i/N, (i = 0, \ldots, N)$. This implies that, when adding some points, all the other points have to be regenerated.

3.2. **Method proposed by Hartinger and Kainhofer.** They propose (see [6]) an inversion type transformation that is also shown to generate G-distributed low-discrepancy sequences.

Theorem 8. *(see [6]) Let $P = (x_1, x_2, \ldots, x_N)$ be a sequence in $[0, 1]$. Consider a continuous type distribution on $[0, 1]$, with density g and distribution function G. Assume that the distribution function G is invertible and $M = \sup_{x\in[0,1]} g(x) < \infty$. Define for $k = 1, \ldots, N$*

$$x_k^- = \max_{\mathcal{A}=\{x_i\in P|G(x_i)<x_k\}} x_i$$

$$x_k^+ = \min_{\mathcal{B}=\{x_i\in P|x_k\leq G(x_i)\}} x_i.$$

Set $x_k^- = 0$ if $\mathcal{A} = \emptyset$ and $x_k^+ = 1$ if $\mathcal{B} = \emptyset$.
Then the G-discrepancy of any transformed sequence $(y_1, y_2, \ldots, y_N)$ with the property that $y_k \in (x_k^-, x_k^+]$ for all $1 \leq k \leq N$ is bounded by

$$D_{N,G}(y_1, \ldots, y_N) \leq (1 + 2M) D_N(x_1, \ldots, x_N).$$

In the method proposed by Hartinger and Kainhofer, any value in the interval $(x_k^-, x_k^+]$ can be considered as $G^{-1}(x_k)$. They do not analyze the possibility of approximating G^{-1} using interpolation methods. Thus, in their method, the kind of interpolation is not relevant for the discrepancy bound and the smoothness of the interpolation is not taken into account.

4. Inversion method using linear Lagrange interpolation

Next, we propose an inversion type method that can be used to generate one-dimensional low-discrepancy sequences with an arbitrary distribution function G. The method is based on the approximation of the inverse of the distribution function by linear Lagrange interpolation. We also give bounds for the G-discrepancy of the generated sequence. Our method is based on the following idea:

Let $0 \leq x_1 < \ldots < x_N \leq 1$ be a one-dimensional sequence. Let G be an invertible distribution function. We define $x_0 = 0$ and $x_{N+1} = 1$. To approximate $G^{-1}(x_k)$, we proceed as follows. First, we determine the interval $(x_i, x_{i+1}]$, with $i \in \{0, 1, \ldots, N\}$ such that $G^{-1}(x_k) \in (x_i, x_{i+1}]$, based on the following equivalence:

$$G^{-1}(x_k) \in (x_i, x_{i+1}], \quad \text{iff} \quad G(x_i) < x_k \leq G(x_{i+1}).$$

Then, we approximate $G^{-1}(x_k)$ with a value y_k in the interval $(x_i, x_{i+1}]$, which is calculated using linear Lagrange interpolation of G^{-1}.

Before describing our method, we recall a lemma from Niederreiter [10].

Lemma 9. *Let* $P_1 = (u_1, \ldots, u_N)$ *and* $P_2 = (v_1, \ldots, v_N)$ *be two sequences in* $[0, 1]$. *If, for all* $1 \leq n \leq N$, *the following condition takes place*

$$|u_n - v_n| \leq \varepsilon$$

then

$$|D_N(u_1, \ldots, u_N) - D_N(v_1, \ldots, v_N)| \leq 2\varepsilon. \tag{4}$$

To prove the main theorems of this paper, we formulate and prove the following results.

Proposition 10. *Let* $(x_1, x_2, \ldots, x_N)$ *be a one-dimensional sequence in* $[0, 1]$, *with* $x_0 := 0 \leq x_1 < x_2 < \ldots < x_N \leq 1 =: x_{N+1}$. *The following inequality takes place*

$$|x_n - x_{n+1}| \leq D_N(x_1, \ldots, x_N), \qquad n = 0, \ldots, N. \tag{5}$$

Proof. We note that

$$|x_n - x_{n+1}| = \left| \frac{1}{N} + \left(\frac{n}{N} - x_n \right) - \left(\frac{n+1}{N} - x_{n+1} \right) \right| = \left| \frac{1}{N} + r_n - r_{n+1} \right|$$

where $r_n = \frac{n}{N} - x_n$, $n = 0, \ldots, N+1$.

It follows that

$$|x_n - x_{n+1}| \leq \max_{\substack{0 \leq i \leq N \\ 1 \leq j \leq N+1}} \left| \frac{1}{N} + r_i - r_j \right| = D_N(x_1, \ldots, x_N).$$

In the last equality, we used the result from Theorem 2. □

Lemma 11. *Consider a continuous type distribution on* $[0,1]$*, with density* g *and distribution function* G*. Assume that the distribution function* G *is invertible and* $g(t) \neq 0, \quad \forall t \in [0,1]$*. If* $G \in C^4([0,1])$ *then* $G^{-1} \in C^4([0,1])$ *and the derivatives of* G^{-1} *have the following expressions:*

$$\begin{aligned}
(G^{-1})' &= \frac{1}{g(G^{-1})} \\
(G^{-1})'' &= -\frac{g'(G^{-1})}{(g(G^{-1}))^3} \\
(G^{-1})^{(3)} &= -\frac{g''(G^{-1})g(G^{-1}) - 3(g'(G^{-1}))^2}{(g(G^{-1}))^5} \\
(G^{-1})^{(4)} &= -\frac{g'''(G^{-1})g^2(G^{-1}) - 10g''(G^{-1})g'(G^{-1})g(G^{-1}) + 15(g'(G^{-1}))^3}{(g(G^{-1}))^7}.
\end{aligned}$$

Their norms are given by:

$$||(G^{-1})''||_\infty = \left|\left| \frac{g'}{g^3} \right|\right|_\infty$$

$$||(G^{-1})^{(4)}||_\infty = \left|\left| \frac{g'''g^2 - 10g''g'g + 15g'^3}{g^7} \right|\right|_\infty.$$

Proof. The proof is immediately. □

Theorem 12 (Lagrange interpolated inversion method). *Let* $0 \leq x_1 < \ldots < x_N \leq 1$ *be a one-dimensional sequence. We consider a continuous type distribution on* $[0,1]$*, with density* g *and distribution function* G*. We assume that the distribution function* G *is invertible,* $\sup_{t\in[0,1]} g(t) \leq M < \infty$ *and* $g(t) \neq 0, \quad \forall t \in [0,1]$*. For each point* x_k*,* $k = 1, \ldots, N$*, we determine the interval* $(x_i, x_{i+1}]$ *such that*

$$G(x_i) < x_k \leq G(x_{i+1}).$$

We denote by $(x_k^-, x_k^+]$ *the determined interval* $(x_i, x_{i+1}]$ *.*

We set $x_k^- = 0$ *if* $x_k \leq G(x_1)$ *and* $x_k^+ = 1$ *if* $G(x_N) < x_k$.

We generate the sequence $(y_1, \ldots, y_N)$ *with*

$$y_k = \frac{x_k - G(x_k^+)}{G(x_k^-) - G(x_k^+)} x_k^- + \frac{x_k - G(x_k^-)}{G(x_k^+) - G(x_k^-)} x_k^+, \qquad k = 1, \ldots, N. \tag{6}$$

If $G \in C^2[0,1]$ *and* $\left\|\frac{g'}{g^3}\right\|_\infty \leq L$, *then the G-discrepancy of the sequence* $(y_1, y_2, \ldots, y_N)$ *is bounded by*

$$D_{N,G}(y_1, \ldots, y_N) \leq \left(1 + M^3 L\right) D_N(x_1, \ldots, x_N). \tag{7}$$

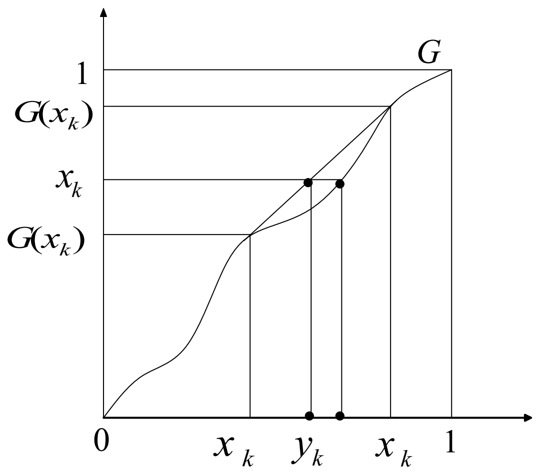

FIGURE 1. Inversion method.

Proof. First, we illustrate how we obtained the values y_k given by (6).

We consider a linear Lagrange interpolation of G^{-1}, with nodes $G\left(x_k^-\right)$ and $G\left(x_k^+\right)$. The values of G^{-1} at the nodes are $G^{-1}(G(x_k^-)) = x_k^-$ and $G^{-1}(G(x_k^+)) = x_k^+$. The interpolation formula is:

$$G^{-1} = L_1 G^{-1} + R_1 G^{-1},$$

where L_1G^{-1} is the Lagrange interpolation polynomial of degree 1 and R_1G^{-1} is the remainder. Using the expression of the Lagrange interpolation polynomial, we get

$$(L_1G^{-1})(x_k) = \frac{x_k - G(x_k^+)}{G(x_k^-) - G(x_k^+)} x_k^- + \frac{x_k - G(x_k^-)}{G(x_k^+) - G(x_k^-)} x_k^+ = y_k.$$

Next, we prove inequality (7). For this, we use the result from Theorem 6 and we obtain

$$\begin{aligned} D_{N,G}(y_1, \ldots, y_N) &= D_{N,G}\big(G^{-1}(G(y_1)), \ldots, G^{-1}(G(y_N))\big) \\ &= D_N\big(G(y_1), \ldots, G(y_N)\big). \end{aligned}$$

It follows that

$$\big|D_{N,G}(y_1, \ldots, y_N) - D_N(x_1, \ldots, x_N)\big| = \big|D_N(G(y_1), \ldots, G(y_N)) - D_N(x_1, \ldots, x_N)\big|. \tag{8}$$

Our intention is to apply Lemma 9 with $P_1 = (x_1, \ldots, x_N)$ and $P_2 = (G(y_1), \ldots, G(y_N))$. For this, we first estimate $|G(y_k) - x_k|$, for $1 \leq k \leq N$, as follows

$$\big|G(y_k) - x_k\big| = \big|G(y_k) - G(G^{-1}(x_k))\big| = \left|\int_{G^{-1}(x_k)}^{y_k} g(t)dt\right| \leq M\big|G^{-1}(x_k) - y_k\big|. \tag{9}$$

We use the bound for the interpolation error (see [2])

$$\big|G^{-1}(x_k) - y_k\big| = \big|R_1(G^{-1})(x_k)\big| \leq \frac{|u(x_k)|}{2!}\big|\big|(G^{-1})''\big|\big|_\infty \tag{10}$$

where

$$|u(x_k)| = \big|\left(x_k - G(x_k^-)\right)\left(x_k - G(x_k^+)\right)\big|. \tag{11}$$

Considering the fact that $G(x_k^-) < x_k \leq G(x_k^+)$, we get

$$\big|x_k - G(x_k^-)\big| \leq \big|G(x_k^+) - G(x_k^-)\big| = \left|\int_{x_k^-}^{x_k^+} g(t)dt\right| \leq M\big|x_k^+ - x_k^-\big|. \tag{12}$$

Since $[x_k^-, x_k^+]$ is an interval of type $[x_i, x_{i+1}]$, we apply Proposition 10 and we get

$$|x_k^+ - x_k^-| = |x_{i+1} - x_i| \leq D_N(x_1, \ldots, x_N).$$

Relation (12) becomes

$$|x_k - G(x_k^-)| \leq M D_N(x_1, \ldots, x_N). \tag{13}$$

In a similar way, we obtain

$$\left|x_k - G(x_k^+)\right| \leq M D_N(x_1, \ldots, x_N). \tag{14}$$

From (11), (13), (14), it follows that

$$|u(x_k)| = \left|\left(x_k - G(x_k^-)\right)\left(x_k - G(x_k^+)\right)\right| \leq M^2 D_N^2(x_1, \ldots, x_N). \tag{15}$$

Using (15) and the result from Lemma 11, relation (10) becomes

$$\left|G^{-1}(x_k) - y_k\right| \leq \frac{M^2 D_N^2(x_1, \ldots, x_N)}{2} \left|\left|\frac{g^{'}}{g^3}\right|\right|_\infty \leq \frac{M^2 D_N^2(x_1, \ldots, x_N)}{2} L. \tag{16}$$

Replacing (16) into (9) we obtain

$$|G(y_k) - x_k| \leq \frac{M^3 D_N^2(x_1, \ldots, x_N)}{2} L, \qquad k = 1, \ldots, N.$$

Applying Lemma 9 with $P_1 = (x_1, \ldots, x_N)$, $P_2 = (G(y_1), \ldots, G(y_N))$, $\varepsilon = \frac{M^3 D_N^2(x_1, \ldots, x_N)}{2} L$ and using $D_N^2 \leq D_N$, as $D_N \leq 1$, we get

$$\left|D_N\left(G(y_1), \ldots, G(y_N)\right) - D_N(x_1, \ldots, x_N)\right| \leq 2\varepsilon \leq M^3 L D_N(x_1, \ldots, x_N). \tag{17}$$

From (8) and (17), we obtain

$$|D_{N,G}(y_1, \ldots, y_N) - D_N(x_1, \ldots, x_N)| \leq M^3 L D_N(x_1, \ldots, x_N).$$

The final result is

$$D_{N,G}(y_1,\ldots,y_N) \le \big(1+M^3L\big)D_N(x_1,\ldots,x_N).$$

□

5. Inversion method using cubic Hermite interpolation

Next, we propose a transformation where the inverse of the distribution function G is approximated using cubic Hermite interpolation. We also give bounds for the G-discrepancy of the generated sequence.

Theorem 13 (Hermite interpolated inversion method). *On the same conditions as in Theorem 12, we consider the sequence* $(y_1,\ldots,y_N)$ *generated by*

$$y_k = h_{00}(x_k)x_k^- + h_{10}(x_k)x_k^+ + h_{01}(x_k)\frac{1}{g(x_k^-)} + h_{11}(x_k)\frac{1}{g(x_k^+)} \tag{18}$$

where

$$\begin{aligned}
h_{00}(x_k) &= \frac{\big(x_k - G(x_k^+)\big)^2}{\big(G(x_k^-)-G(x_k^+)\big)^2}\left(1-2\frac{x_k-G(x_k^-)}{G(x_k^-)-G(x_k^+)}\right)\\
h_{10}(x_k) &= \frac{\big(x_k - G(x_k^-)\big)^2}{\big(G(x_k^+)-G(x_k^-)\big)^2}\left(1-2\frac{x_k-G(x_k^+)}{G(x_k^+)-G(x_k^-)}\right)\\
h_{01}(x_k) &= \frac{\big(x_k - G(x_k^-)\big)\big(x_k - G(x_k^+)\big)^2}{\big(G(x_k^-)-G(x_k^+)\big)^2}\\
h_{11}(x_k) &= \frac{\big(x_k - G(x_k^+)\big)\big(x_k - G(x_k^-)\big)^2}{\big(G(x_k^+)-G(x_k^-)\big)^2}
\end{aligned}$$

for all $k=1,\ldots,N$.

If $G \in C^4[0,1]$ *and* $\left\|\frac{g'''g^2-10g''g'g+15g'^3}{g^7}\right\|_\infty \le L$, *then the* G*-discrepancy of the sequence* $(y_1,\ldots,y_N)$ *is bounded by*

$$D_{N,G}(y_1,\ldots,y_N) \le \left(1+\frac{M^5L}{12}\right)D_N(x_1,\ldots,x_N). \tag{19}$$

Proof. First, we explain how we generated the values y_k given by (18).

We consider a cubic Hermite interpolation of G^{-1} with double nodes $G(x_k^-)$ and $G(x_k^+)$. The values of G^{-1} and $(G^{-1})'$ at the nodes are $G^{-1}(G(x_k^-)) = x_k^-$, $G^{-1}(G(x_k^+)) = x_k^+$, $(G^{-1})'(G(x_k^-)) = \frac{1}{g(x_k^-)}$ and $(G^{-1})'(G(x_k^+)) = \frac{1}{g(x_k^+)}$.

The Hermite interpolation formula is:

$$G^{-1} = H_3G^{-1} + R_3G^{-1},$$

where H_3G^{-1} is the Hermite interpolation polynomial of degree 3 and R_3G^{-1} is the remainder. Using the expression of the Hermite polynomial with double nodes (see [2]), it can be proved, after some calculus, that

$$(H_3G^{-1})(x_k) = y_k.$$

Next, we follow the same steps as in Theorem 12. We point out only the differences. The bound for the interpolation error (see [2]) is given by

$$\left|R_3(G^{-1})(x_k)\right| = \left|G^{-1}(x_k) - y_k\right| \leq \frac{|u(x_k)|}{4!}\left|\left|(G^{-1})^{(4)}\right|\right|_\infty \tag{20}$$

where

$$|u(x_k)| = \left|\left(x_k - G\left(x_k^-\right)\right)^2\left(x_k - G(x_k^+)\right)^2\right| \leq M^4D_N^4(x_1, \ldots, x_N).$$

We use the result from Lemma 11

$$\left|\left|(G^{-1})^{(4)}\right|\right|_\infty = \left|\left|\frac{g'''g^2 - 10g''g'g + 15g'^3}{g^7}\right|\right|_\infty \leq L.$$

Relation (20) becomes

$$|G^{-1}(x_k) - y_k| \leq \frac{M^4D_N^4(x_1, \ldots, x_N)}{4!}L.$$

Similar to Theorem 12, we obtain

$$|G(y_k) - x_k| \leq \frac{M^5L}{24}D_N^4(x_1, \ldots, x_N).$$

Applying Lemma 9 with $\varepsilon = \frac{M^5 L}{24} D_N^4(x_1, \ldots, x_N)$ and using $D_N^4 \leq D_N$, as $D_N \leq 1$, we get

$$|D_N(G(y_1), \ldots, G(y_N)) - D_N(x_1, \ldots, x_N)| \leq 2\varepsilon \leq \frac{M^5 L}{12} D_N(x_1, \ldots, x_N). \qquad (21)$$

Similar to Theorem 12, this implies that

$$|D_{N,G}(y_1, \ldots, y_N) - D_N(x_1, \ldots, x_N)| \leq \frac{M^5 L}{12} D_N(x_1, \ldots, x_N). \qquad (22)$$

The final result is

$$|D_{N,G}(y_1, \ldots, y_N)| \leq \left(1 + \frac{M^5 L}{12}\right) D_N(x_1, \ldots, x_N). \qquad (23)$$

□

The inversion method using linear Lagrange interpolation or cubic Hermite interpolation can be used to G-distributed low discrepancy sequences. The generation of low-discrepancy sequences with an arbitrary distribution function G is described in the Algorithm 14.

Algorithm 14. *Inversion method using interpolation*

Input data: the uniformly distributed low-discrepancy sequence $(x_1, \ldots, x_N)$;

for $k = 1, \ldots, N$ **do**

Find the values x_k^- and x_k^+;

Calculate the point y_k;

end for

Output data: the G-distributed low-discrepancy sequence $(y_1, \ldots, y_N)$.

The method that we proposed uses some values of G, g and derivatives of g to approximate G^{-1}. This is an advantage, as for some distributions the expression of G^{-1} is not known. Note that, in our method, adding one point to the generated sequence would not change the other elements of the sequence, which is another advantage.

References

[1] Chelson, P., *Quasi-Random Techniques for Monte Carlo Methods*, Ph. D. Dissertation, The Claremont Graduate School, 1976.

[2] Coman, Gh., *Numerical Analysis*, Ed. Libris, Cluj-Napoca, 1995 (in Romanian).

[3] Faure, H., *Discrépances de suites associées à un système de numération* (en dimension un), Bull. Soc. Math. France, 109 (1981), 143-182.

[4] Faure, H., *Discrépances de suites associées à un système de numération* (en dimension s), Acta Arith., 41 (1982), 337-351.

[5] Halton, J. H., *On the efficiency of certain quasi-random sequences of points in evaluating multidimensional integrals*, Numer. Math, 2 (1960), 84-90.

[6] Hartinger, J., Kainhofer, R., *Non-uniform low-discrepancy sequence generation and integration of singular integrands*, Submitted, 2004.

[7] Hlawka, E., *Gleichverteilung und Simulation*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II, 206 (1997), 183-216.

[8] Hlawka, E., Mück, R., *A transformation of equidistributed sequences*, In Applications of number theory to numerical analysis, Academic Press, New York, 1972, 371-388.

[9] Hlawka, E., Mück, R., *Über Eine Transformation von gleichverteilten Folgen*, II, Computing, 9 (1972), 127-138.

[10] Niederreiter, H., *Random number generation and Quasi-Monte Carlo methods*, volume 63 of SIAM Conf. Ser. Appl. Math. SIAM, Philadelphia, 1992.

[11] Okten, G., *Error Reduction Techniques in Quasi-Monte Carlo Integration*, Mathematical and Computer Modelling, Vol. 30, No 7-8 (1999), 61-69.

[12] Owen, A. B., *Multidimensional variation for Quasi-Monte Carlo*, Technical report, Standford University, 2004.

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