STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 2, June 2005

ON A FIRST-ORDER NONLINEAR DIFFERENTIAL SUBORDINATION II

GH. OROS AND GEORGIA IRINA OROS

Abstract. We find conditions on the complex-valued functions A, B, C, D defined in the unit disc U and the positive constants M and N such that

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

1. **Introduction and preliminaries**

In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where B, C, D and h are complex-valued functions.

A more general version of (1) is given by

$$B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{2}$$

where $\Omega \subseteq \mathbb{C}$.

In this paper we shall extend this problem by considering a first-order nonlinear differential subordination given by

$$A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z) \prec h(z). \tag{3}$$

A more general version of (3) is given by:

$$A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z) \in \Omega, \tag{4}$$

Received by the editors: 06.05.2005.

2000 *Mathematics Subject Classification.* 30C80.

Key words and phrases. Differential subordination, dominant.

where $\Omega \subseteq \mathbb{C}$.

The general problem is to find conditions on the complex-valued functions A, B, C, D and h such that the differential subordination given by (3) or (4) will have dominants and even best dominant.

We let U denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C};\ |z| < 1\}, \quad \overline{U} = \{z \in \mathbb{C};\ |z| \leq 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in U,\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$$

and

$$A_n = \{f \in U,\ f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$$

and $A_1 = A$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U :\ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Lemma A. [1] (Lemma 2.2.d p.24) *Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

be analytic in U with $p(z) \not\equiv a$, and $n \geq 1$.

If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ and $\zeta \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$

(i) $p(z_0) = q(\zeta)$

(ii) $z_0 p'(z_0) = m\zeta q'(\zeta)$, *and*

(iii) $\mathrm{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \mathrm{Re}\ \left[\dfrac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right]$.

In this paper we consider the first-order nonlinear differential subordination (4) in which $\Omega = \{w;\ |w| < M\}$. Given the functions A, B, C, D and the constant

M, our problem is to find a constant N such that, for $p \in \mathcal{H}[0, n]$, the differential inequality

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies

$$|p(z)| < N.$$

If $D(0) = 0$, then this result can be written in terms of the differential subordination as

$$A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z) \prec Mz$$

implies $p(z) \prec Nz$.

2. **Main results**

In this paper we improve the results obtained in [2].

Theorem 1. *Let $M > 0$, $N > 0$ and let n be a positive integer. Suppose that the functions $A, B, C, D : U \to \mathbb{C}$ satisfy*

$$n|A(z)| - |C(z)| \geq \frac{M + N^2|B(z)| + |D(z)|}{N}. \tag{5}$$

If $p \in \mathcal{H}[0, n]$ and

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M \tag{6}$$

then

$$|p(z)| < N, \quad z \in U.$$

Proof. If we let

$$w(z) = A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z),$$

then from (6) we obtain

$$|w(z)| = |A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)|. \tag{7}$$

From (7) and (6) we have

$$|w(z)| < M, \quad z \in U. \tag{8}$$

Assume that $|p(z)| \not< N$, which is equivalent with $p(z) \not\prec Nz = q(z)$.

According to Lemma A, with $q(z) = Nz$, there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$, $\theta_0 \in [0, 2\pi)$, $\zeta \in \partial U$, $|\zeta| = 1$ and $m \geq n$, such that $p(z_0) = N\zeta$ and $z_0 p'(z_0) = mN\zeta$.

Using these conditions in (3) we obtain for $z = z_0$

$$|w(z_0)| = |A(z_0)z_0 p'(z_0) + B(z_0)p^2(z_0) + C(z_0)p(z_0) + D(z_0)| \tag{9}$$

$$= |A(z_0)mN\zeta + B(z_0)N^2\zeta^2 + C(z_0)N\zeta + D(z_0)|$$

$$\geq |A(z_0)mN + B(z_0)N^2\zeta + C(z_0)N| - |D(z_0)|$$

$$\geq N|A(z_0)m + C(z_0)| - N^2|B(z_0)| - |D(z_0)|$$

$$\geq mn|A(z_0)| - N|C(z_0)| - N^2|B(z_0)| - |D(z_0)|$$

$$\geq [n|A(z_0)| - |C(z_0)|]N - N^2|B(z_0)| - |D(z_0)| \geq M.$$

Since (9) contradicts (8) we obtain the desired results $|p(z)| < N$. □

Instead of prescribing the constant N in Theorem 1, in some cases we can use in (5) to determine an appropriate $N = N(M, n, A, B, C, D)$ so that (6) implies $|p(z)| < N$. This can be accomplished by solving (5) for N and by taking the supremum of the resulting function over U. The condition (5) is equivalent to

$$N^2|B(z)| - N[n|A(z)| - |C(z)|] + |D(z)| + M \leq 0. \tag{10}$$

Suppose $B(z) \neq 0$, the inequality (10) holds if

$$[n|A(z)| - |C(z)|]^2 \geq 4|B(z)|[|D(z)| + M]. \tag{11}$$

The roots of the trinomial in (10) are

$$N_{1,2} = \frac{n|A(z)| - |C(z)| \pm \sqrt{[n|A(z)| - |C(z)|]^2 - 4|B(z)|[|D(z)| + M]}}{2|B(z)|}.$$

Let

$$N = \sup_{|z|<1} \frac{n|A(z)| - |C(z)| - \sqrt{[n|A(z)| - |C(z)|]^2 - 4|B(z)|[|D(z)| + M]}}{2|B(z)|}$$

$$= \sup_{|z|<1} \frac{2[|D(z)| + M]}{n|A(z)| - |C(z)| + \sqrt{[n|A(z)| - |C(z)|]^2 - 4|B(z)|[|D(z)| + M]}}.$$

If this supremum is finite, we have the following version of the Theorem 1:

Theorem 2. *Let $M > 0$, $N > 0$ and n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and the functions $A, B, C, D : U \to \mathbb{C}$, with $B(z) \neq 0$, satisfy:*

$$[n|A(z)| - |C(z)|]^2 \geq 4|B(z)|[|D(z)| + M].$$

$$N = \sup_{|z|<1} \frac{2[|D(z)| + M]}{n|A(z)| - |C(z)| + \sqrt{[n|A(z)| - |C(z)|]^2 - 4|B(z)|[|D(z)| + M]}}$$

then

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies

$$|p(z)| < N, \quad z \in U.$$

If $D(z) \equiv 0$, the Theorem 1 can be rewritten as the following:

Corollary 1. *Let $M > 0$, $N > 0$ and n be a positive integer. Suppose that the functions $A, B, C : U \to \mathbb{C}$ satisfy*

$$n|A(z)| - |C(z)| \geq \frac{M + N^2|C(z)|}{N}.$$

If $p \in \mathcal{H}[0, n]$ and

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

then

$$|p(z)| < N, \quad z \in U.$$

References

[1] Miller, S. S., Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Oros, Gh., Oros, Georgia Irina, *On a first-order nonlinear differential subordination I*, Analele Univ. Oradea, Fasc. Matematică, Tome IX, 5-12, 2002, 65-70.

Department of Mathematics, University of Oradea
Str. Armatei Române, No. 5, 410087 Oradea, Romania
E-mail address: georgia_oros_ro@yahoo.co.uk