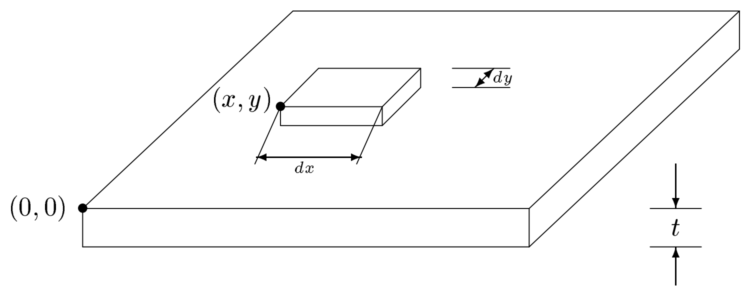
STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 2, June 2005

ON SOME DISTRIBUTION PROBLEM OF THE TEMPERATURE IN A METALLIC PLATE

CĂTĂLIN MITRAN

Abstract. Many problems of technical, industrial, economical type have can be simulated using the differential equations or the partial equations derivatives. But many times the determining of an analytical solution is a difficult, even impossible problem. This is the reason for which the numerical approximation, generally, and in this case the finite differences are a good solution for solving the mentioned problems.

We shall consider a plate made uniformly which has a thickness of t and who contain an element of measure $dx \times dy$. We shall take u the independent variable who represent the temperature into the element. The location of the element is in (x, y), situated in the left position of the plate. We shall consider that the high tide cross the element on the positive x axis direction and also cross the element in the y direction too as it can be seen in the next figure:



Received by the editors: 13.10.2004.
2000 *Mathematics Subject Classification*. 35Q05, 35Q72.
Key words and phrases. Partial differential equations and applications.

The ratio in which the heat divides the element in the x direction is given by

$$-kA\frac{\partial u}{\partial x} = -k(tdy)\frac{\partial u}{\partial x} \tag{1}$$

and, similar, in the y direction is given by

$$-kA\frac{\partial u}{\partial y} = -k(tdx)\frac{\partial u}{\partial y} \tag{2}$$

where k is the conductivity and A is the area.

We have that the ratio of the high tide who cross in must be equal with the ratio of the high tide who cross out. The high tide who cross in is given by

$$-kA\frac{\partial u}{\partial x} - kA\frac{\partial u}{\partial y} = -k(tdy)\frac{\partial u}{\partial x} - k(tdx)\frac{\partial u}{\partial y} \tag{3}$$

and the high tide who cross out is given by

$$-k(tdy)\left[\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}dx\right] - k(tdx)\left[\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\right] + Q(dxdy) \tag{4}$$

We shall obtain that

$$kt\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(dxdy) = Q(dxdy) \tag{5}$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{Q}{kt} \tag{6}$$

where Q is the heat.

If the object is considered as being in the space the relation (6) will become

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{Q}{kt} \tag{7}$$

that means

$$\Delta u = \frac{Q}{kt}. \tag{8}$$

If the thickness of the plate is variable with x and y the relation (7) become

$$t\Delta^2 u + \frac{\partial t}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial t}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{Q}{k} \tag{9}$$

If the thickness and the thermic conductivity of the plate are variable the relation (8) will become

$$kt\Delta^2 u + \left(k\frac{\partial t}{\partial x} + t\frac{\partial k}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \left(k\frac{\partial t}{\partial y} + t\frac{\partial k}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right) = Q \tag{10}$$

We shall have the next approximation:

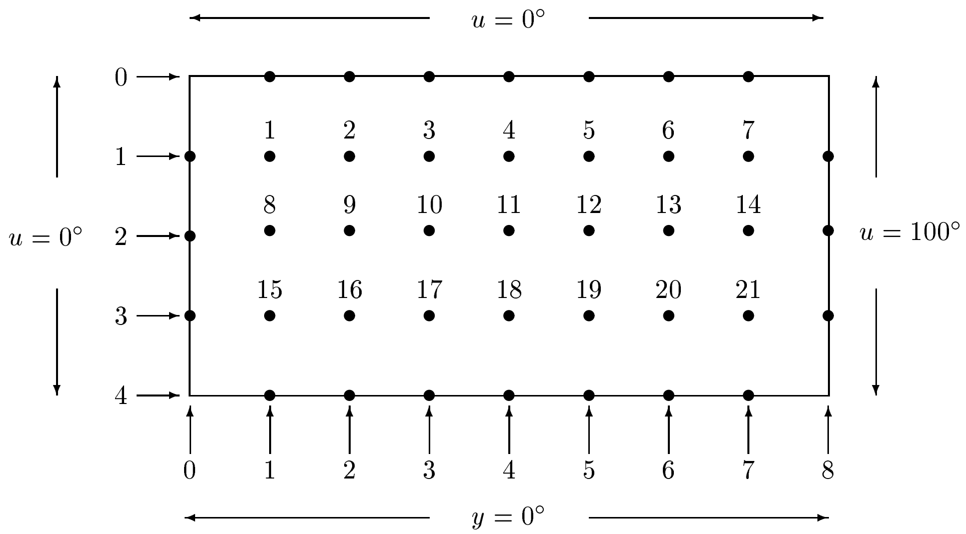
$$\frac{\partial^2 u}{\partial x^2} = \frac{u_L - 2u_0 + u_R}{(\Delta x)^2}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{u_U - 2u_0 + u_D}{(\Delta y)^2} \tag{11}$$

where u_L and u_R are the temperatures at the left and right and u_U and u_D are the temperatures up and down of the considered knot.

Usual, we have that $\Delta x = \Delta y = h$ and from this we shall obtain that

$$\Delta^2 u = \frac{u_L + u_R + u_U + u_D - 4u_0}{h^2} \tag{12}$$

We shall present an example who will show how can be applied the presented formulas with finite differences. Let us consider the next problem where the dates are presented in the next figure:



So, we have a plate of dimensions 20 cm × 10 cm and the space between the knots is about 2.5 cm. We have 21 interior knots. On three sides of the plate the value of u is 0°. On the last side the value of u is 100°. If we suppose that $Q = 0$ the equation (6) will be

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If we take care of (10) and (11) we shall have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_L - 2u_0 + u_R}{2.5^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_U - 2u_0 + u_R}{2.5^2}$$

respectively

$$\frac{u_L + u_R + u_U + u_D - 4u_0}{2.5^2} = 0.$$

We shall write the expression (12) in the case of a lot of knots. We shall have:

1) for the knot no. 1: $-4u_1 + u_2 + u_8 = 0$;

2) for the knot no. 7: $u_6 - 4u_7 + u_{14} = -100$;

3) for the knot no. 9: $u_2 + u_8 - 4u_9 + u_{10} + u_{16} = 0$;

4) for the knot no. 19: $u_{18} + u_{12} - 4u_{19} + u_{20} = 0$;

5) for the knot no. 21: $u_{20} + u_{14} - 4u_{21} = -100$.

Finally, we shall obtain a system of 21 equations with a number of 21 unknowns with the solution given by the next table:

Line	Column No. 1	Column No. 2	Column No. 3	Column No. 4	Column No. 5	Column No. 6	Column No. 7
1	0.3530	0.9132	2.0103	4.2957	9.1531	19.6631	43.2101
1	0.4988	1.2894	2.8323	7.0193	12.6537	27.2893	53.1774
3	0.3530	0.9132	2.0103	4.2957	9.1531	19.6631	43.2101

Many other techniques of solving the above mentioned equations use iterative methods in which a knot is written depending of other knots. For example we shall have the relations:

$$u_{i,j} = \frac{i_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{4}$$

or

$$u_{i,j} = \frac{u_{i-1,j-1} + 4u_{i,j-1} + u_{i+1,j} + 4u_{i,j-1} - 20u_{i,j} + 4u_{i,j+1} + u_{i+1,j-1} + 4u_{i+1,j} + u_{i+1,j+1}}{6}.$$

Definition 1. An equation of type:

$$\Delta u = R$$

where $R = R(x,y)$ is a function defined on the same domain with u will be named just a *Poisson equation.*

For example, if we have to solve the next equation:

$$\Delta^2 u = -2$$

we shall use the next approximation

$$u_{i,j} = \frac{u_{i,j-1} + u_{i,j+1} + u_{i-1,j} + u_{i+1,j} + 2}{4}.$$

Definition 2. An equation of type $\Delta u = 0$ and which satisfies a condition of the next type:

$$Au + B = Cu' \tag{13}$$

where A, B, C are constants will be named an *equation of Neumann type.*

Remark 1.

$$-ku' = H(u - u_s) \tag{14}$$

is also a condition of Neumann type if we take care of (12) and take $A = H$, $B = -Hu_s$, $C = -k$.

We shall present a new model with a plate having the dimensions 5 cm × 9 cm and a thickness of 0.5 cm. We will take the next values:

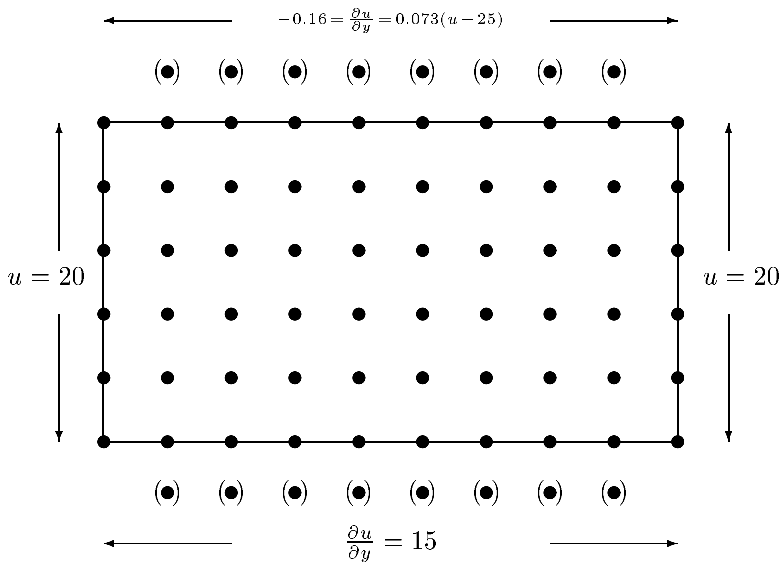
1) $h = 1$ cm;

2) $Q = 0.6 cal/s/cm^3$;

3) $k = 0.16$ is the thermic conductivity;

4) $H = 0.073$ is the coefficient of the heat transfer.

The frontier conditions are given by the next figure:



Solving the problem using the mentioned approximations we shall obtain the next values included in the next table:

20.000	73.510	107.915	128.859	138.826	138.826	128.859	107.915	73.510	20.000
20.000	99.195	137.476	167.733	180.743	180.743	167.733	137.476	99.195	20.000
20.000	99.793	155.061	189.855	207.669	207.669	189.855	155.061	99.793	20.000
20.000	103.918	163.119	200.956	219.410	219.410	200.956	163.119	103.918	20.000
20.000	102.762	162.539	201.442	220.604	220.604	201.442	162.539	102.762	20.000
20.000	94.589	152.834	191.669	210.959	210.959	191.669	152.834	94.589	20.000

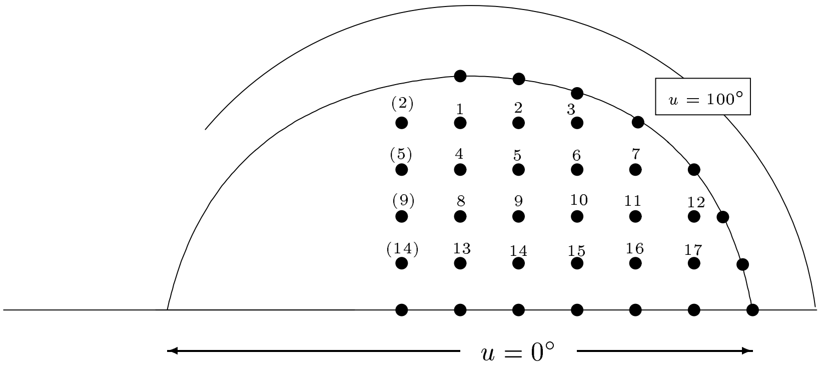
If we have a circular domain or, generally, an irregular domain it is recommended to use the polar coordinates:

$$\Delta^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{15}$$

and, from here, the approximation formula

$$\Delta^2 u = \frac{u_L - 2u_0 + u_R}{(\Delta r)^2} + \left(\frac{1}{r}\right)\left(\frac{u_R - u_L}{2\Delta r}\right) + \frac{1}{r^2}\left(\frac{u_U - 2u_0 + u_D}{(\Delta\theta)^2}\right) \tag{16}$$

Using (16) we shall solve the problem whose dates are given in the next figure:



We shall obtain the results presented in the next table:

Knot	Calculated value	Analytical value
1	86.053	85.906
2	87.548	87.417
3	92.124	92.094
4	69.116	68.807
5	70.733	70.482
6	75.994	765.772
7	85.471	85.405
8	48.864	48.448
9	50.436	50.000
10	55.606	55.151
11	65.891	65.593
12	84.189	84.195
13	25.466	25.133
14	27.501	27.109
15	30.102	29.527
16	38.300	37.436
17	57.206	57.006

References

[1] Heatley, G., *Numerical analysis with applications*, Addison Wesley Longman, Inc., U.S.A., 1999.

[2] Dragotă, I., Petrehuş, V., *Metode numerice pentru ecuaţii diferenţiale*, Editura Orizonturi Universitare, Timişoara, 2002.

[3] Arieh Iserles, *Numerical analysis of differential equations*, Cambridge Univ. Press, USA, 1996.

Department of Mathematics, University of Petroşani