STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

EXTREMAL PROBLEMS OF TURÁN TYPE

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Abstract. We give estimations of certain weighted L^2-norms of the k-th derivative of polynomials which have a curved majorant. They are obtained as applications of special quadrature formulae.

1. **Introduction**

The following problem was raised by P. Turán.

Let $\varphi(x) \geq 0$ *for* $-1 \leq x \leq 1$ *and consider the class* $P_{n,\varphi}$ *of all polynomials of degree* n *such that* $|p_n(x)| \leq \varphi(x)$ *for* $-1 \leq x \leq 1$.

How large can $max_{[-1,1]} \left|p_n^{(k)}(x)\right|$ *be if* p_n *is arbitrary in* $P_{n,\varphi}$ *?*

The aim of this paper is to consider the solution in the weighted L^2-norm for the majorant

$$\varphi(x) = \frac{\alpha - \beta x}{\sqrt{1-x^2}}, 0 \leq \beta \leq \alpha.$$

Let us denote by

$$x_i = \cos\frac{(2i-1)\pi}{2n}, i = 1, 2, ..., n, \text{ the zeros of } T_n(x) = \cos n\theta, x = \cos\theta, \tag{1.1}$$

the Chebyshev polynomial of the first kind,

$$y_i^{(k)} \text{ the zeros of } U_{n-1}^{(k)}(x), U_{n-1}(x) = \sin n\theta / \sin\theta, x = \cos\theta, \tag{1.2}$$

the Chebyshev polynomial of the second kind and

$$G_{n-1}(x) = \alpha U_{n-1}(x) - \beta U_{n-2}(x), 0 \leq \beta \leq \alpha. \tag{1.3}$$

Received by the editors: 14.03.2005.

2000 *Mathematics Subject Classification.* 41A17, 41A05, 41A55, 65D30.

Key words and phrases. Bouzitat quadrature, Chebyshev polynomials, curved majorant.

Let $\Pi_{\alpha,\beta}$ be the class of all polynomials p_{n-1}, of degree $\leq n-1$ such that

$$|p_{n-1}(x_i)| \leq \frac{\alpha - \beta x_i}{\sqrt{1-x_i^2}}, i = 1, 2, ..., n, \tag{1.4}$$

where the x_i's are given by (1.1) and $0 \leq \beta \leq \alpha$.

2. **Results**

Theorem 2.1. *If* $p_{n-1} \in \Pi_{\alpha,\beta}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \tag{2.1}$$

$$\leq \frac{2\pi n (n-1) \left[\left(\alpha^2+\beta^2\right)(n-2)\left(n^2-2n+2\right) + 5\beta^2\left(n^2-n+1\right)\right]}{15}$$

with equality for $p_{n-1} = G_{n-1}$.

Two cases are of special interest:

I. Case $\alpha = \beta = 1$, $\varphi(x) = \sqrt{\frac{1-x}{1+x}}$,

$G_{n-1}(x) = V_{n-1}(x) = \frac{\cos\left[\left(n-\frac{1}{2}\right)\arccos x\right]}{\cos\left[\frac{1}{2}\arccos x\right]}$.

Note that $P_{n-1,\varphi} \subset \Pi_{1,1}$, $V_{n-1} \notin P_{n-1,\varphi}$, $V_{n-1} \in \Pi_{1,1}$.

Corollary 2.2. *If* $p_{n-1} \in \Pi_{1,1}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \frac{2\pi n(n-1)(2n-1)\left(n^2-n+3\right)}{15} \tag{2.2}$$

with equality for $p_{n-1} = V_{n-1}$.

II. Case $\alpha = 1$, $\beta = 0$, $\varphi(x) = \frac{1}{\sqrt{1-x^2}}$, $G_{n-1} = U_{n-1}$.

Note that $P_{n-1,\varphi} \subset \Pi_{1,0}$, $U_{n-1} \in P_{n-1,\varphi}$, $U_{n-1} \in \Pi_{1,0}$.

Corollary 2.3. *If* $p_{n-1} \in \Pi_{1,0}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \frac{2\pi n\left(n^4-1\right)}{15} \tag{2.3}$$

with equality for $p_{n-1} = U_{n-1}$.

In this second case we have a more general result:

Theorem 2.4. *If* $p_{n-1} \in \Pi_{1,0}$ *and* $0 \leq b \leq a$ *then we have*

$$\int_{-1}^{1} (a-bx)^3 \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.4}$$

$$\leq \frac{\pi a (n+k+1)!}{(n-k-2)!} \left[\frac{2\left(n^2-(k+2)^2\right)\left(a^2+3b^2\right)}{(2k+1)(2k+3)(2k+5)} + \frac{2(k+1)a^2+3b^2}{(2k+1)(2k+3)} \right]$$

$k = 0, ..., n-2$, *with equality for* $p_{n-1} = U_{n-1}$.

Setting $a = 1, b = 1$ one obtains the following

Corollary 2.5. *If* $p_{n-1} \in \Pi_{1,1}$ *then we have*

$$\int_{-1}^{1} (1-x)^{k+5/2} (1+x)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.5}$$

$$\leq \frac{\pi (n+k+1)!}{(n-k-2)!} \times \frac{8\left(n^2-(k+2)^2\right)+(2k+5)^2}{(2k+1)(2k+3)(2k+5)}$$

$k = 0, ..., n-2$, *with equality for* $p_{n-1} = U_{n-1}$.

Setting $a = 1, b = 0$ one obtains the following

Corollary 2.6. *If* $p_{n-1} \in \Pi_{1,0}$ *then we have*

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.6}$$

$$\leq \frac{2\pi (n+k+1)!}{(n-k-2)!} \times \frac{n^2+k^2+3k+1}{(2k+1)(2k+3)(2k+5)}$$

$k = 0, ..., n-2$, *with equality for* $p_{n-1} = U_{n-1}$.

3. Lemmas

Here we state some lemmas which help us in proving our theorems.

Lemma 3.1. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \leq \frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}}, i = 1, 2, ..., n,$ *where the* x_i*'s are given by (1.1). Then we have*

$$\left|p'_{n-1}(y_j)\right| \leq \left|G'_{n-1}(y_j)\right|, \; k = 0, 1, ..., n-1, \textit{and} \tag{3.1}$$

$$\left|p'_{n-1}(1)\right| \leq \left|G'_{n-1}(1)\right|, \; \left|p'_{n-1}(-1)\right| \leq \left|G'_{n-1}(-1)\right| \; . \tag{3.2}$$

Proof. By the Lagrange interpolation formula based on the zeros of T_n and using $T_n'(x_i) = \frac{(-1)^{i+1}n}{(1-x_i^2)^{1/2}}$, we can represent any polynomial p_{n-1} by $p_{n-1}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_n(x)}{x-x_i}(-1)^{i+1}(1-x_i^2)^{1/2} p_{n-1}(x_i)$.

From $G_{n-1}(x_i) = (-1)^{i+1}\frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}}$ we have $G_{n-1}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{T_n(x)}{x-x_i}(\alpha-\beta x_i)$.

Differentiating with respect to x we obtain

$p_{n-1}'(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{T_n'(x)(x-x_i)-T_n(x)}{(x-x_i)^2}(-1)^{i+1}(1-x_i^2)^{1/2}p_{n-1}(x_i)$.

On the roots of $T_n'(x) = nU_{n-1}(x)$ and using (1.4) we find

$\left|p_{n-1}'(y_j)\right| \le \frac{1}{n}\sum_{i=1}^{n}\frac{|T_n(y_j)|}{(y_j-x_i)^2}(\alpha-\beta x_i) = \frac{|T_n(y_j)|}{n}\sum_{i=1}^{n}\frac{\alpha-\beta x_i}{(y_j-x_i)^2} = \left|G_{n-1}'(y_j)\right|$.

For $l_i(x) = \frac{T_n(x)}{x-x_i}$ taking into account that $l_i'(1) > 0$ (see [5]) it follows

$\left|p_{n-1}'(1)\right| \le \frac{1}{n}\sum_{i=1}^{n} l_i'(1)(\alpha-\beta x_i) = \left|G_{n-1}'(1)\right|$.

Similarly $\left|p_{n-1}'(-1)\right| \le \left|G_{n-1}'(-1)\right|$. □

Lemma 3.2. $(Duffin - Schaeffer)$[2] *If* $q(x) = c\prod_{i=1}^{n}(x-x_i)$ *is a polynomial of degree* n *with* n *distinct real zeros and if* $p \in P_n$ *is such that*

$$|p'(x_i)| \le |q'(x_i)| \quad (i = 1, 2, ..., n),$$

then for $k = 1, 2, ..., n-1$,

$$\left|p^{(k+1)}(x)\right| \le \left|q^{(k+1)}(x)\right| \text{ whenever } q^{(k)}(x) = 0.$$

Lemma 3.3. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \frac{1}{\sqrt{1-x_i^2}}, i = 1, 2, ..., n,$ *where the* x_i*'s are given by (1.1). Then we have*

$$\left|p_{n-1}^{(k+1)}(y_j^{(k)})\right| \le \left|U_{n-1}^{(k+1)}(y_j^{(k)})\right|, \text{ whenever } U_{n-1}^{(k)}(y_j^{(k)}) = 0, \tag{3.3}$$

$$k = 0, 1, ..., n-1, and$$

$$\left|p_{n-1}^{(k+1)}(1)\right| \le \left|U_{n-1}^{(k+1)}(1)\right|, \left|p_{n-1}^{(k+1)}(-1)\right| \le \left|U_{n-1}^{(k+1)}(-1)\right| . \tag{3.4}$$

Proof. For $\alpha = 1$, $\beta = 0$, $G_{n-1} = U_{n-1}$ and (3.1) give $\left|p_{n-1}'(y_j)\right| \le \left|U_{n-1}'(y_j)\right|$ and (3.2) $\left|p_{n-1}'(1)\right| \le \left|U_{n-1}'(1)\right|$, $\left|p_{n-1}'(-1)\right| \le \left|U_{n-1}'(-1)\right|$.

Now the proof ends by applying Duffin-Schaeffer Lemma. □

We need the following quadrature formulae:

Lemma 3.4. *For any given* n *and* k, $0 \le k \le n-1$, *let* $y_i^{(k)}$, $i = 1, ..., n-k-1$, *be the zeros of* $U_{n-1}^{(k)}$.

Then the quadrature formulae

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} f(x)\,dx = A_0\left[f(-1)+f(1)\right] + \sum_{i=1}^{n-k-1} s_i f\left(y_i^{(k)}\right), \tag{3.5}$$

$$A_0 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!}{(n+k)!}, s_i > 0$$

and

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} f(x)\,dx = B_0\left[f(-1)+f(1)\right] \tag{3.6}$$

$$+C_0\left[f'(-1)-f'(1)\right] + \sum_{i=1}^{n-k-2} v_i f\left(y_i^{(k+1)}\right)$$

$$C_0 = \frac{2^{2k}(2k+3)\,\Gamma(k+3/2)^2\,(n-k-2)!}{(n+k+1)!},$$

$$B_0 = C_0\frac{2\left(n^2-(k+2)^2\right)(2k+3)+4(k+1)(2k+5)}{(2k+1)(2k+5)}$$

have algebraic degree of precision $2n-2k-1$.

For $r(x) = (a-bx)^3$, $0 \le b \le a$ *the formulae*

$$\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} f(x)\,dx = A_1 f(-1) + B_1 f(1) \tag{3.7}$$

$$+\sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right) f\left(y_i^{(k)}\right)$$

$$A_1 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!\,(a+b)^3}{(n+k)!},$$

$$B_1 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!\,(a-b)^3}{(n+k)!}$$

and

$$\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} f(x)\,dx = C_1 f(-1) + D_1 f(1) \tag{3.8}$$

$$+C_2 f'(-1) - D_2 f'(1) + \sum_{i=1}^{n-k-2} v_i r\left(y_i^{(k+1)}\right) f\left(y_i^{(k+1)}\right),$$

$$C_1 = B_0 (a+b)^3 - 3C_0 b (a+b)^2, D_1 = B_0 (a-b)^3 + 3C_0 b (a-b)^2,$$

$$C_2 = C_0 (a+b)^3, D_2 = C_0 (a-b)^3,$$

have algebraic degree of precision $2n-2k-4$.

Proof. The first quadrature formula (3.5) is the Bouzitat quadrature formula of the second kind [3, *formula* (4.8.1)], for the zeros of $U_{n-1}^{(k)} = cP_{n-k-1}^{\left(k+\frac{1}{2},k+\frac{1}{2}\right)}$.
Setting $\alpha = \beta = k-1/2, m = n-k-1$ in [3, *formula* (4.8.5)] we find A_0 and $s_i > 0$ (cf. [3, *formula* (4.8.4)]).

If in the above quadrature formula (3.6), we put

$$f(x) = (1-x)(1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

$$U_{n-1}^{(k+1)}(x) = cP_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

we obtain C_0, and for

$$f(x) = (1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x)$$

we find B_0.
If in formula (3.5) we replace $f(x)$ with $r(x) f(x)$ we get (3.7) and
if in formula (3.6) we replace $f(x)$ with $r(x) f(x)$ we get (3.8). □

4. **Proof of the Theorems**

Proof of Theorem 2.1

Setting $k=0$ in (3.5) we find the formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2n}\left[f(-1)+f(1)\right] + \frac{\pi}{n}\sum_{i=1}^{n-1} f(y_i) \tag{4.1}$$

According to this quadrature formula and using (3.1) and (3.2) we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx = \frac{\pi}{2n} \left(p'_{n-1}(-1)\right)^2 + \frac{\pi}{2n} \left(p'_{n-1}(1)\right)^2 + \frac{\pi}{n} \sum_{i=1}^{n-1} \left(p'_{n-1}(y_i)\right)^2$$

$$\leq \frac{\pi}{2n} \left(G'_{n-1}(-1)\right)^2 + \frac{\pi}{2n} \left(G'_{n-1}(1)\right)^2 + \frac{\pi}{n} \sum_{i=1}^{n-1} \left(G'_{n-1}(y_i)\right)^2 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[G'_{n-1}(x)\right]^2 dx.$$

Using the following formula ($k = 0$ in (3.6))

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{3\pi\left(3n^2-2\right)}{10n(n^2-1)} \left[f(-1) + f(1)\right] + \frac{3\pi}{4n(n^2-1)} \left[f'(-1) - f'(1)\right] + \sum_{i=1}^{n-2} v_i f(y'_i)$$

we find $\int_{-1}^{1} \frac{\left[U'_{n-1}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi n\left(n^4-1\right)}{15}$, $\int_{-1}^{1} \frac{\left[U'_{n-2}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi n(n-1)(n-2)\left(n^2-2n+2\right)}{15}$

and $\int_{-1}^{1} \frac{\left[G'_{n-1}(x)\right]^2}{\sqrt{1-x^2}} dx = \frac{2\pi n(n-1)\left[\left(\alpha^2+\beta^2\right)(n+1)\left(n^2+1\right)-5\beta^2\left(n^2-n+1\right)\right]}{15}$.

Proof of Theorem 2.4

According to the quadrature formula (3.7), positivitiness of s_i's, and using (3.3) and (3.4) we have

$$\int_{-1}^{1} (a-bx)^3 \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx$$

$$= A_1 \left[p_{n-1}^{(k+1)}(-1)\right]^2 + B_1 \left[p_{n-1}^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right) \left[p_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2$$

$$\leq A_1 \left[U_{n-1}^{(k+1)}(-1)\right]^2 + B_1 \left[U_{n-1}^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right) \left[U_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2$$

$$= \int_{-1}^{1} (a-bx)^3 \left(1-x^2\right)^{k-1/2} \left[U_{n-1}^{(k+1)}(x)\right]^2 dx$$

In order to complete the proof we apply formula (3.8) to $f = \left[U_{n-1}^{(k+1)}(x)\right]^2$. Having in mind $U_{n-1}^{(k+1)}\left(y_i^{(k+1)}\right) = 0$ and the following relations deduced from [1]

$$U_{n-1}^{(k+1)}(1) = \frac{n\left(n^2-1^2\right)\ldots\left(n^2-(k+1)^2\right)}{1.3\ldots(2k+3)}, \quad U_{n-1}^{(k+2)}(1) = \frac{n^2-(k+2)^2}{2k+5} U_{n-1}^{(k+1)}(1),$$

$$U_{n-1}^{(k+1)}(-1)\, U_{n-1}^{(k+2)}(-1) = -U_{n-1}^{(k+1)}(1)\, U_{n-1}^{(k+2)}(1),$$

we find

$$\int_{-1}^{1} (a-bx)^3 \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx = C_1 \left[U_{n-1}^{(k+1)}(-1)\right]^2 + D_1 \left[U_{n-1}^{(k+1)}(1)\right]^2$$

$$+2C_2 U_{n-1}^{(k+1)}(-1)\, U_{n-1}^{(k+2)}(-1) - 2D_2 U_{n-1}^{(k+1)}(1)\, U_{n-1}^{(k+2)}(1)$$

$$= \frac{\pi a(n+k+1)!}{(n-k-2)!} \left[\frac{2\left[n^2-(k+2)^2\right]\left(a^2+3b^2\right)}{(2k+1)(2k+3)(2k+5)} + \frac{2(k+1)a^2+3b^2}{(2k+1)(2k+3)}\right].$$

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