STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ASYMPTOTIC PROPERTIES OF THE DISCRETIZED PANTOGRAPH EQUATION

PETR KUNDRÁT

Abstract. We are going to deal with the asymptotic properties of all solutions of the delay difference equation

$$\Delta x_n = -ax_n + bx_{\left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor}, \qquad n = 0, 1, 2, \dots ,$$

where $a > 0$, $b \neq 0$ are reals. This equation represents the discretization of the corresponding delay differential equation. Our aim is to show the resemblance in the asymptotic bounds of solutions of the discrete and continuous equation and discuss some numerical problems connected with this investigation.

1. Introduction

We discuss the numerical discretization of the delay differential equation

$$\dot{x}(t) = -ax(t) + bx(\tau(t)), \qquad t \in I := [t_0, \infty) \tag{1}$$

in the form

$$x_{n+1} - x_n = -ahx_n + bhx_{\tau_n}, \tag{2}$$

where $a > 0$, $b \neq 0$ are reals, $\tau_n := \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$, $t_n := t_0 + nh$, $n = 0, 1, 2, \dots$, $h > 0$ is the stepsize and the symbol $\lfloor\ \rfloor$ is an integer part. Then x_n means the approximation of $x(t_n)$.

Equation (2) is a difference equation obtained from (1) via the modified Euler method. It has been shown in [2] that numerical schema (2) is convergent. Our aim

Received by the editors: 08.11.2004.

2000 *Mathematics Subject Classification.* 39A11, 34K25.

Key words and phrases. Differential equation, difference equation.

is to describe some asymptotic properties of equation (2) (more precisely, to find conditions under which asymptotic behaviour of (1) and (2) is similar).

We especially investigate equations with unbounded lag, i.e. such that $t - \tau(t) \to \infty$ as $t \to \infty$. As the prototype of this equation may serve the so called pantograph equation (i.e. equation (1) with $\tau(t) = qt$, $0 < q < 1$). The name of the equation has its origin in the application on British railways [9], where the motion of pantograph of electric locomotive along trolley wire has been described.

In the connection with the investigation of asymptotic properties of solutions of these equations we recall papers dealing with relative problems, e.g., Čermák [1], Heard [4], Iserles [5], Liu [7], Kato and McLeod [6] and many others in the continuous case, and Györi and Pituk [3], Makay and Terjéki [8], Péics[10] and others in the discrete case.

The paper is organized as follows. In the next section we recall the asymptotic estimate of all solutions of (1) (valid under certain assumptions). In Section 4 we derive the analogous asymptotic estimate valid for all solutions of difference equation (2).

2. **Continuous case**

In this section we mention the result describing the asymptotics of the investigated delay differential equation.

Theorem 2.1 (Heard [4]**).** *Let $a > 0$, $b \neq 0$ be scalars, $\tau \in C^2(I)$ be such that $\dot{\tau}$ is positive and decreasing on I and $q = \dot{\tau}(t_0) < 1$. Then for any solution x of (1) there exists a continuous periodic function g of period $\log q^{-1}$ such that*

$$x(t) = (\varphi(t))^{\alpha}\, g(\log \varphi(t)) + O\left((\varphi(t))^{\alpha_r - 1}\right) \qquad \text{as} \quad t \to \infty,$$

where φ is a solution of

$$\varphi(\tau(t)) = q\varphi(t), \qquad t \in I, \tag{3}$$

$\alpha = \log(b/a) \big/ \log q^{-1}$ and $\alpha_r = \Re(\alpha)$.

Remark 2.2. *Particularly, it follows from Theorem 2.1 that for any solution x of (1) holds*

$$x(t) = O\left(\psi(t)\right) \qquad as \quad t \to \infty,$$

where $\psi(t) = (\varphi(t))^{\alpha_r}$ is a solution of the functional equation

$$a\psi(t) = |b|\psi(\tau(t)), \qquad t \in I. \tag{4}$$

3. **Preliminaries**

In this section we summarize the assumptions necessary to formulate the result for discrete case. First, let us denote (H) the assumptions on function τ:
(H): Let τ be an increasing continuous function on I such that $\tau(t) < t$ for all $t \in I$ (the case $\tau(t_0) = t_0$ is also possible), $\tau(t + \tilde{h}) - \tau(t)$ is nonincreasing for arbitrary $\tilde{h}$ fulfilling $0 < \tilde{h} \leq h$ on I and let $\lim_{t\to\infty} \tau(t) = \infty$.

Further, throughout this paper we denote $T_{-1} = \tau(t_0)$ and $T_k = \tau^{-k}(t_0)$, $k = 0, 1, 2, \ldots$, where τ^{-k} means the k-th iteration of the inverse τ^{-1}. If we set $I_m := [T_{m-1}, T_m]$ for all $m = 0, 1, 2, \ldots$, then τ is mapping I_{m+1} onto I_m.

Instead of the above functional equation (4) we consider the functional inequality

$$a\rho(t) \geq |b|\rho(t_0 + \left\lfloor \frac{\tau(t) - t_0}{h} \right\rfloor h), \qquad t \in I. \tag{5}$$

Now we can formulate the proposition ensuring some required properties of solutions of the inequality (5).

Proposition 3.1. *Consider the inequality (5), where $a > 0$, $b \neq 0$ are reals and let (H) be fulfilled.*

- **(i):** *If $|b|/a \geq 1$, then there exists a positive continuous nondecreasing solution ρ of inequality (5).*
- **(ii):** *If $|b|/a < 1$, then there exists a positive continuous decreasing solution ρ of inequality (5) such that $\rho(t+\tilde{h})-\rho(t)$ is nondecreasing on I for arbitrary real $0 < \tilde{h} \leq h$.*

Proof. Using the step method we can easily verify that there exists a positive continuous solution $\rho(t)$ of (5) which is nondecreasing or decreasing according to $|b|/a \geq 1$ or $|b|/a < 1$, respectively.

Further, we assume that $|b|/a < 1$ and show that the function $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I for all $0 < \tilde{h} \leq h$. It is easy to check that any solution of functional equation

$$a\rho(t) = |b|\rho(\tau(t) - h) \tag{6}$$

is fulfilling the inequality (5). We choose the decreasing function ρ defined on the initial interval I_0 such that $\rho(t_0) = (|b|/a)\rho(T_{-1} - h)$ and let $\rho(t+\tilde{h}) - \rho(t)$ be nondecreasing on I_0. Further let $t^*, t^{**} \in I_1$, $t^* < t^{**}$. If we denote $h^* := \tau(t^* + \tilde{h}) - \tau(t^*)$, $h^{**} := \tau(t^{**} + \tilde{h}) - \tau(t^{**})$, then $h^* \geq h^{**}$ and we can write

$$\begin{aligned}
&\rho(t^* + \tilde{h}) - \rho(t^*) = \tfrac{|b|}{a}\Big(\rho(\tau(t^* + \tilde{h}) - h) - \rho(\tau(t^*) - h)\Big)\\
&= \tfrac{|b|}{a}\Big(\rho(\tau(t^*) + h^* - h) - \rho(\tau(t^*) - h)\Big)\\
&\leq \tfrac{|b|}{a}\Big(\rho(\tau(t^{**}) + h^* - h) - \rho(\tau(t^{**}) - h)\Big)\\
&\leq \tfrac{|b|}{a}\Big(\rho(\tau(t^{**}) + h^{**} - h) - \rho(\tau(t^{**}) - h)\Big)\\
&= \tfrac{|b|}{a}\Big(\rho(\tau(t^{**} + \tilde{h}) - h) - \rho(\tau(t^{**}) - h)\Big) = \rho(t^{**} + \tilde{h}) - \rho(t^{**})
\end{aligned}$$

by use of the assumptions of proposition. Thus $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on $I_0 \cup I_1$ and repeating this procedure for intervals $I_2, I_3, \ldots$ we obtain that the function $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I. □

4. **Main result**

Theorem 4.1. *Let x_n, $n = 0, 1, 2, \ldots$ be a solution of (2), where $0 < ah < 1$, $b \neq 0$ are reals. Let (H) be fulfilled, let ρ be a positive solution of (5) with the properties guaranteed by Proposition 3.1 and let $\rho_n := \rho(t_n)$.*

- **(i):** *If $|b|/a \geq 1$, then $x_n = O(\rho_n)$ as $n \to \infty$.*
- **(ii):** *If $|b|/a < 1$ and moreover*

$$\sum_{k=1}^{\infty} \frac{\rho(T_{k-1}) - \rho(T_{k-1} + h)}{\rho(T_{k+1})} < \infty,$$

then $x_n = O(\rho_n)$ as $n \to \infty$.

Proof. First we rewrite the difference equation (2) as

$$x_{n+1} = \tilde{a}^h x_n + hbx_{\tau_n}, \quad n = 1, 2, 3, \ldots, \tag{7}$$

where $\tilde{a}$ is a (unique) positive real such that $\tilde{a}^h = 1 - ah$.

We introduce the substitution $y_n = x_n/\rho_n$ in (7) to obtain

$$\rho_{n+1} y_{n+1} = \tilde{a}^h \rho_n y_n + bh\rho_{\tau_n} y_{\tau_n}, \quad n = 1, 2, 3, \ldots \tag{8}$$

and show that every solution y_n of (8) is bounded as $n \to \infty$. Multiplying the previous equality by $1/\tilde{a}^{t_n+h}$ we get

$$\frac{\rho_{n+1} y_{n+1}}{\tilde{a}^{t_n+h}} = \frac{\rho_n y_n}{\tilde{a}^{t_n}} + \frac{bh}{\tilde{a}^{t_n+h}} \rho_{\tau_n} y_{\tau_n},$$

i.e.,

$$\Delta\left(\frac{\rho_n y_n}{\tilde{a}^{t_n}}\right) = \frac{bh}{\tilde{a}^{t_n+h}} \rho_{\tau_n} y_{\tau_n}. \tag{9}$$

Now we take any $\bar{t} \in I_{m+1}$, $m = 1, 2, \ldots$. We define nonnegative integers $k_m(\bar{t}) := \lfloor (\bar{t} - T_m)/h \rfloor$. Denote $\bar{t}_m := \bar{t} - k_m(\bar{t})h - h$. Summing the equation (9) from $\bar{t}_m$ to $\bar{t} - h$, we get

$$y(\bar{t}) = \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} y(\bar{t}_m) + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \frac{bh}{\tilde{a}^{s+h}} \rho_{\tau_s} y_{\tau_s}.$$

Let us denote $M_m := \sup\left\{ |y(t)|, \quad t \in \bigcup_{j=0}^{m} I_j \right\}$. In accordance with (5) we obtain

$$|y(\bar{t})| \le \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} M_m + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \frac{(1 - \tilde{a}^h)\rho_s}{\tilde{a}^{s+h}} M_m.$$

Using the relation $\frac{(1-\tilde{a}^h)}{\tilde{a}^{s+h}} = \Delta\left(\frac{1}{\tilde{a}}\right)^s$ we get

$$|y(\bar{t})| \le \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} M_m + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \left(\Delta\left(\frac{1}{\tilde{a}}\right)^s\right) \rho_s M_m$$

and summing by parts we finally have

$$\begin{aligned} |y(\bar t)| &\le M_m \left\{ \frac{\rho(\bar t_m)\tilde a^{\bar t}}{\rho(\bar t)\tilde a^{\bar t_m}} + \frac{\tilde a^{\bar t}}{\rho(\bar t)} \left(\frac{\rho(\bar t)}{\tilde a^{\bar t}} - \frac{\rho(\bar t_m)}{\tilde a^{\bar t_m}} - \sum_{s=\bar t_m}^{\bar t - h} \left(\frac{1}{\tilde a}\right)^{s+h} \Delta\rho_s \right) \right\} \\ &= M_m \left\{ 1 - \frac{\tilde a^{\bar t}}{\rho(\bar t)} \sum_{s=\bar t_m}^{\bar t - h} \left(\frac{1}{\tilde a}\right)^{s+h} \Delta\rho_s \right\}. \end{aligned} \tag{10}$$

The common part of the proof ends here and we continue for the cases (i) and (ii) separately.

ad (i): If $|b|/a \ge 1$, then in accordance with Proposition 3.1 we choose a non-decreasing function $\rho(t)$ on I. Then $\Delta\rho(t)$ is nonnegative on I, hence $|y(\bar t)| \le M_m$. Since $\bar t \in I_{m+1}$ was arbitrary, we have $M_{m+1} \le M_m$, i.e., M_m is bounded as $m \to \infty$. Hence the function $y(t)$ is bounded and the statement (i) is proved.

ad (ii): If $|b|/a < 1$, then in accordance with Proposition 3.1 we choose a decreasing function $\rho(t)$ on I such that $\Delta\rho(t)$ is nondecreasing on I. Then from (10) we have

$$\begin{aligned} |y(\bar t)| &\le M_m \left\{ 1 + \frac{\rho(\bar t_m) - \rho(\bar t_m + h)}{\rho(\bar t)} \sum_{s=\bar t_m}^{\bar t - h} \left(\frac{\tilde a^{\bar t}}{\tilde a^{s+h}} \right) \right\} \\ &\le M_m \left\{ 1 + \frac{\rho(\bar t_m) - \rho(\bar t_m + h)}{\rho(\bar t)} \xi \right\} \le M_m \left\{ 1 + \xi \frac{-\Delta\rho(T_m - h)}{\rho(T_{m+1})} \right\}, \end{aligned}$$

where $\xi := 1/(1 - \tilde a^h)$. The repeated application of this procedure yields

$$|y(\bar t)| \le M_1 \prod_{j=1}^{m} \left(1 + \xi \frac{-\Delta\rho(T_j - h)}{\rho(T_{j+1})} \right),$$

i.e.,

$$M_{m+1} \le M_1 \prod_{j=1}^{m} \left(1 + \xi \frac{-\Delta\rho(T_j - h)}{\rho(T_{j+1})} \right).$$

By our assumption, the product converges as $m \to \infty$, hence M_m is bounded as $m \to \infty$. The theorem is proved. □

Remark 4.2. The assumption on the stepsize h ($h < 1/a$) enables us to preserve the correlation of asymptotic estimates of discrete and continuous case. In other words, the estimates of solutions in the discrete case and the continuous case are expressed via the same function, resp. sequence (provided the stepsize h is sufficiently small).

Remark 4.3. In the estimate concerning the case $|b|/a < 1$ it is also possible to take a solution ψ of functional equation (4) instead of a function ρ (which is a solution of (5)). Using the fact that the term $\rho(\tau(t)) - \rho(\tau(t) - h)$ is a positive nonincreasing function it could be shown that there exists a solution ψ of (4) such that $\psi(t) > \rho(t)$ for all $t > t_0$. In some cases the utilizing of ψ instead of ρ can be more applicable.

5. Examples

Corollary 5.1. Consider the scalar pantograph equation

$$\dot{x}(t) = -ax(t) + bx(qt), \tag{11}$$

where $a > 0$, $b \neq 0$, $0 < q < 1$ are reals. The qualitative theory yields the estimate

$$x(t) = O(t^r), \qquad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}}, \qquad \text{as} \quad t \to \infty \tag{12}$$

for every solution x of the equation (11). The corresponding difference equation is

$$x_{n+1} = (1 - ah)x_n + bhx_{\lfloor \frac{qt_n - t_0}{h} \rfloor}, \qquad t \geq t_0 > 0, \tag{13}$$

where the above assumptions on a, b, q are fulfilled and $0 < ah < 1$. Then the following estimate

$$x_n = O(t_n^r), \quad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}} \qquad \text{as} \quad n \to \infty \tag{14}$$

is valid for all solutions $\{x_n\}_{n=0}^{\infty}$ of difference equation (13).

Example 5.2. Consider the initial problem:

$$\dot{x}(t) = -2x(t) + x(t/2), \quad x(0) = 1, \quad t \in [0, \infty)\,. \tag{15}$$

In accordance with (12) we get the asymptotic estimate

$$x(t) = O(1/t) \qquad \text{as} \quad t \to \infty.$$

In the corresponding discrete case we consider formula (13) in the form

$$x_{n+1} = (1 - 2h)x_n + hx_{\lfloor t_n/2h \rfloor}, \qquad t_0 = 0, \qquad x_0 = 1. \tag{16}$$

Then $x_n = O(1/t_n)$ as $t \to \infty$ provided $h < 1/a$. If we violate the condition on stepsize, this asymptotic formula is not valid. Indeed, if $h = 1 > 1/a$, then the corresponding discrete equation $x_{n+1} = -x_n + x_{\lfloor t_n/2 \rfloor}$ admits solutions not tending to zero as $n \to \infty$.

It is obvious, that the assumption $0 < ah < 1$ has its relevance in the choice of suitable stepsize h to preserve the same behaviour of difference case as in the continuous case.

References

[1] Čermák, J., *The asymptotics of solutions for a class of delay differential equations*, Rocky Mountain J. Mathematics, 33 (2003), pp. 775–786.

[2] Feldstein, A., *Discretization methods for retarded ordinary differential equation*, Ph.D. Thesis, Dept. of Mathematics, UCLA, Los Angeles, 1964.

[3] Györi, I., Pituk, M., *Comparison theorems and asymptotic equilibrium for delay differential and difference equations*, Dynam. Systems Appl., 5 (1996), pp. 277-302.

[4] Heard, M. L., *A change of variables for functional differential equations*, J. Differential Equations, 18 (1975), pp. 1-10.

[5] Iserles, A., *Exact and discretized stability of the pantograph equation*, Appl. Numer. Math., 24 (1997), pp. 295-308.

[6] Kato, T., McLeod, J. B., *The functional differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc., 77 (1971), pp. 891-937.

[7] Liu, Y., *Asymptotic behaviour of functional-differential equations with proportional time delays*, European J. Appl. Math., 7 (1996), pp. 11-30.

[8] Makay, G., Terjéki, J., *On the asymptotic behavior of the pantograph equations*, Electron J. Qual. Theory Differ. Equ., 2 (1998), pp. 1-12, (electronic).

[9] Ockendon, J. R., Tayler, A. B., *The dynamics of a current collection system for an electric locomotive*, Proc. Roy. Soc. Lond. A., 322 (1971), pp. 447–468.

[10] Péics, H., *On the asymptotic behaviour of difference equations with continuous arguments*, Ser A. Math. Anal., 9(2) (2002), pp. 257-273.

Institute of Mathematics, FME BUT, Technická 2, 616 69 Brno, Czech Republic

E-mail address: kundrat@centrum.cz