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ON THE NUMERICAL SIMULATION OF A LOW-MACH NUMBER FLOW

I. GASSER, J. STRUCKMEIER, AND IOAN TELEAGA

Abstract. In the present work we investigate numerically a flow model used to simulate convection problems such as tunnel fires. This model is based on an asymptotic approach for Navier-Stokes equations first derived in [2]. We will show that this model is capable to combine the low-Mach number limit with large temperature gradients. Two sets of calculations are included in this work to show the capabilities of the proposed model and also the usefulness of the standard Boussinesq approximation.

1. Introduction

Because of many fire accidents in tunnels, the interest in the description, modeling and the simulation of such events has been increased in the last years. In practice, to simulate a complete fire accident is not possible due to many parameters involved: the tunnel geometry, the number of cars inside, the intensity and position of the fire, ventilation rules etc. In time two main features of fire events have been observed, namely characteristic velocities in the tunnel of the order of $1\,m/s$ and characteristic temperature differences which are quite large [1].

In [2], [3] and [4] a mathematical model which combines these two features has been developed and numerically tested. The modeling starts with the description of the air flow using the compressible Navier-Stokes equations. Then, using appropriate scales

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(see [2]), the two-dimensional compressible system is written as:

$$
\begin{aligned}
(\rho)_t + div(\rho \boldsymbol{u}) &= 0 \\
\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + (\gamma M^2)^{-1}\frac{1}{\rho}\nabla p &= \frac{1}{\rho}\left(Re^{-1}\Delta \boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right) + \boldsymbol{f} \\
(\rho T)_t + div(\boldsymbol{u}\rho T) + (\gamma - 1)p\,div(\boldsymbol{u}) &= \gamma Pr^{-1}Re^{-1}\Delta T + \boldsymbol{q}
\end{aligned} \tag{1}
$$

where $\rho, \boldsymbol{u}, p$, and T represent the density, the velocity field, the pressure and the temperature, respectively. The functions $\boldsymbol{f}$, $\boldsymbol{q}$ are the external force (e.g gravity) and the heat source due to the fire which acts as a volume indicator function over the fire. The dimensionless constants γ, M, Re, Pr and Fr are the adiabatic exponent, the Mach number, the Reynolds number and the Prandtl number, respectively. All these quantities and reference values are detalied in [2].

Since $M \ll 1$, a compressible flow solver will suffer severe deficiencies, both in efficiency and accuracy. Two distinct techniques have been proposed to capture solution convergence for low-Mach number flows: preconditioning and asymptotic expansion methods. In fact these techniques rescale the condition number of the system. The first one is to multiply time derivatives by suitable preconditioning matrix, in the sense that they scale the eigenvalues of the system to similar orders of magnitude and remove the disparity in wave speeds, leading to a well-conditioned system [5].

In this work, we will follow the second technique, the asymptotic or perturbation method. This approach consists in a Taylor series expansion of variables (in our case the pressure) in power terms of the Mach number. The basic philosophy behind this technique is to decrease the numerical representation of the speed of sound artificially, by substracting a constant pressure p_0 across the entire domain:

$$p = p_0 + (\gamma M^2)p_1 + O((\gamma M^2)^2),$$

where p_0 is the ground pressure and p_1 is the fluctuation pressure part. It turns out that the ground pressure can be only a function of time, i.e $p_0 = p_0(t)$, but since the tunnel is a open domain this ground pressure will also not change in time. Therefore, considering $p_0 = constant$ and that in leading order we have $T = p_0/\rho$, the system

(1) can be rewritten as [2]

$$
\begin{aligned}
(\rho)_t + div(\rho\boldsymbol{u}) &= 0 && (2)\\
\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p_1 &= \frac{1}{\rho}\left(Re^{-1}\Delta\boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right) + \boldsymbol{f} && (3)\\
div(\boldsymbol{u}) &= \gamma Pr^{-1}Re^{-1}\Delta(\frac{1}{\rho}) + \frac{q}{\gamma p_0}. && (4)
\end{aligned}
$$

This system represents a density-dependent flow with a non-vanishing divergence of the velocity field.

The system (2)-(4) is solved numerically by a modified first order projection method described in [3]. For the numerical scheme we prescribe the following boundary conditions

$$
\begin{aligned}
\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}(\boldsymbol{x},t) &= 0, \quad \boldsymbol{x}\in\Gamma_1\cup\Gamma_3\\
\boldsymbol{u}(\boldsymbol{x},t) &= 0, \quad \boldsymbol{x}\in\Gamma_2\cup\Gamma_4\\
\rho(\boldsymbol{x},t) &= \rho_0, \quad if\, \boldsymbol{u}(\boldsymbol{x},t) > 0, \quad \boldsymbol{x}\in\Gamma_1\\
\rho(\boldsymbol{x},t) &= \rho_1, \quad if\, \boldsymbol{u}(\boldsymbol{x},t) < 0, \quad \boldsymbol{x}\in\Gamma_3\\
p(\boldsymbol{x},t) &= p_0, \quad \boldsymbol{x}\in\Gamma_1\\
p(\boldsymbol{x},t) &= p_1, \quad \boldsymbol{x}\in\Gamma_3\\
\nabla p\cdot\vec{n} &= \frac{1}{\rho}\left(Re^{-1}\Delta\boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right)\cdot\vec{n} + \boldsymbol{f}\cdot\vec{n}, \quad \boldsymbol{x}\in\Gamma_2\cup\Gamma_4
\end{aligned}
$$

where Γ_1, Γ_3 denote the entrance and the exit of the tunnel and Γ_2, Γ_4 the lower and upper fixed walls, respectively.

2. The validity of the Boussinesq approximation in the case of large temperature differences

The Boussinesq approximation starts by considering the compressible Navier-Stokes equations for fluid flow. At this stage all fluid properties are assumed to be functions of temperature T and presssure P, i.e.

$$\rho = \rho(T,P), \qquad c_p = c_p(T,P)$$
$$\mu = \mu(T,P), \qquad \alpha = \alpha(T,P)$$
$$k = k(T,P)$$

Because these functions are not known completely, one assumes that each function may be approximated by a first order Taylor expansion, i.e.

$$\begin{aligned}
\rho &= \rho_r(1-\alpha_r(T-T_r)+\beta_r(P-P_r)) \\
c_p &= c_{p_r}(1+a_r(T-T_r)+b_r(P-P_r)) \\
\mu &= \mu_r(1+c_r(T-T_r)+d_r(P-P_r)) \\
\alpha &= \alpha_r(1+e_r(T-T_r)+f_r(P-P_r)) \\
k &= k_r(1+m_r(T-T_r)+n_r(P-P_r))
\end{aligned} \tag{5}$$

with $\boldsymbol{x}_r = (\rho_r, c_{p_r}, \mu_r, \alpha_r, k_r)$ where $w_r = (\alpha_r, a_r, c_r, e_r, m_r)$ represents the reference states of $(1/\boldsymbol{x}_r)\partial \boldsymbol{x}_r/\partial T$ and $\boldsymbol{y}_r = (\beta_r, b_r, d_r, f_r, n_r)$ represents the reference states of $(1/\boldsymbol{x}_r)\partial \boldsymbol{x}_r/\partial P$, respectively.

According to [6] the following criteria must be checked in order to ensure the validity of the Boussinesq approximation:

$$c_1 = |\alpha_r\theta| \le 0.1, \qquad c_2 = |\beta_r\rho_r\boldsymbol{g}L| \le 0.1 \tag{6}$$

$$c_3 = |c_r\theta| \le 0.1, \qquad c_4 = |d_r\rho_r\boldsymbol{g}L| \le 0.1 \tag{7}$$

$$c_5 = |a_r\theta| \le 0.1, \qquad c_6 = |b_r\rho_r\boldsymbol{g}L| \le 0.1 \tag{8}$$

$$c_7 = |m_r\theta| \le 0.1, \qquad c_8 = |n_r\rho_r\boldsymbol{g}L| \le 0.1 \tag{9}$$

$$c_9 = |e_r\theta| \le 0.1, \qquad c_{10} = |f_r\rho_r\boldsymbol{g}L| \le 0.1 \tag{10}$$

$$c_{11} = |\frac{\alpha_r\boldsymbol{g}L}{c_{p_0}}| \le 0.1 \qquad c_{12} = |\frac{\alpha_r\boldsymbol{g}LT_r}{c_{p_0}\theta}| \le 0.1 \tag{11}$$

$$c_{13} = |\frac{\alpha_r \boldsymbol{g} L}{c_{p_0}}|(Pr\, Ra^{-1})^{1/2} \quad \leq \quad 0.1(Pr\, Ra)^{-1/2} \tag{12}$$

where θ, $\boldsymbol{g}$, L are the maximal temperature variations around T_r, the gravitational force and the reference length, respectively. In the case of air at $T_r = 15\,°C$ and $P_r = 10^5 Pa$ the following values for the criteria $c_1 - c_{11}$ are given in [6]:

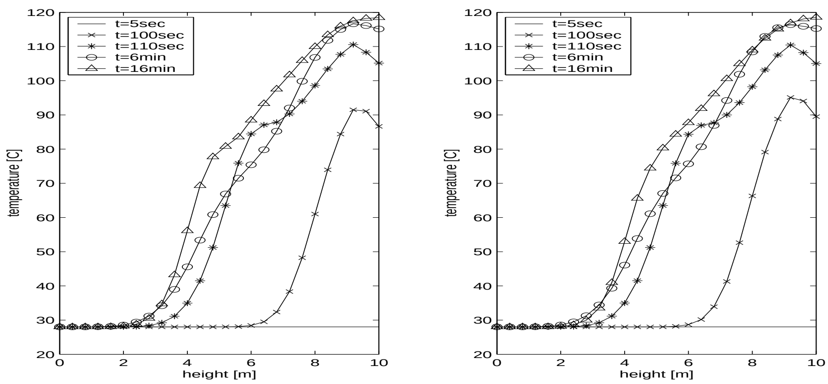
c_1	c_2	c_3	c_4	c_5	c_6
$3.5 \cdot 10^{-3}\theta$	$1.2 \cdot 10^{-6} L$	$2.8 \cdot 10^{-3}\theta$	0	$4.5 \cdot 10^{-5}\theta$	$2.3 \cdot 10^{-9} L$

c_7	c_8	c_9	c_{10}	c_{11}
$2.4 \cdot 10^{-3}\theta$	0	$-3.6 \cdot 10^{-3}\theta$	0	$3.6 \cdot 10^{-7} L$

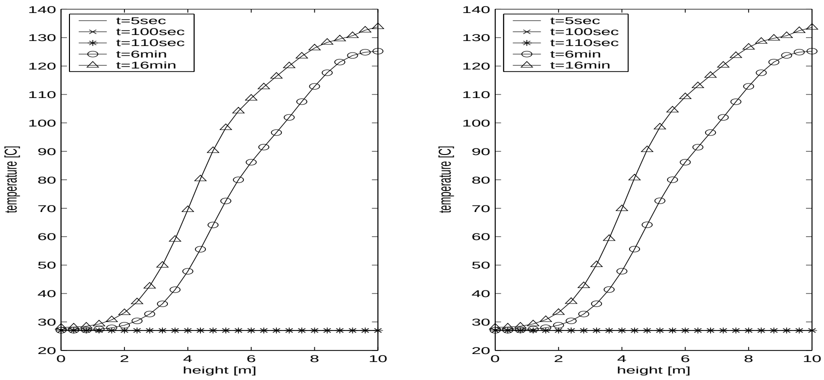
If the maximal temperature difference θ is very large (e.g. $1000\,°C$) then it is quite easy to check that the criteria c_1, c_3, c_7, c_9 are not fullfiled, hence the Boussinesq approximation does not apply.

3. Numerical results

In the following we will compare the numerical results in the case of two realistic tunnel fire events described in [3] with the Boussinesq approximation [6]. In both cases the heat source is placed exactly in the middle of the tunnel and it is distributed over a rectangular area of size 10 m x 4 m.



(a)



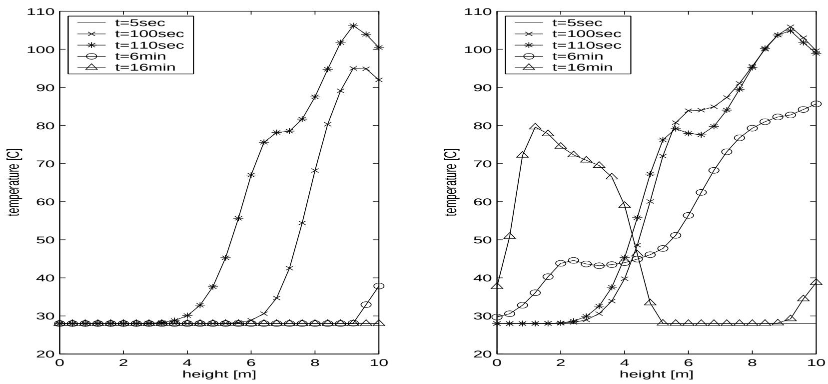
(b)

Figure 1. Vertical temperature profiles (in °C) for a tunnel without slope 100 m left and right from the heat source at various times: (a) the low-Mach number model (1), (b) the standard incompressible Navier-Stokes model with Boussinesq approximation.

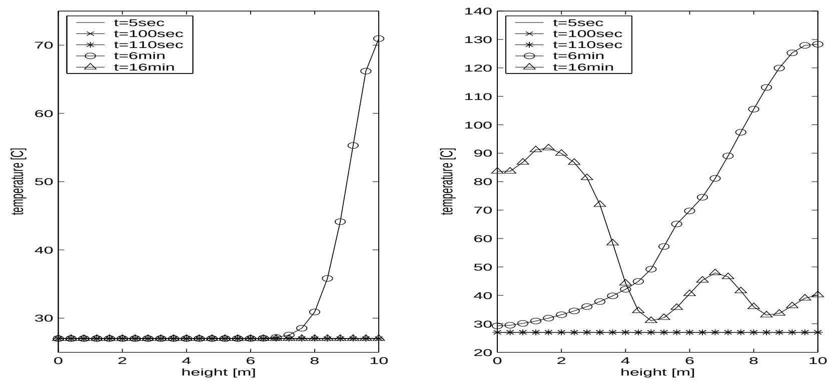
3.1. **Tunnel without slope.** The tunnel configuration data are listed in Table I. More information about the numerical method and other relevand data are given in [3]. Figure 1 show the temperature profiles along a vertical axis, which is placed 100 m to the left and right of the middle of the tunnel in the case of the low-Mach number model (1) (1a), and the standard Boussinesq approximation model (1b). First of all the results show that the flow field is symmetric with respect to the location of the heat source.

Table I. Test configuration

Length	$1000m$
Height	$10m$
Heat source	$1MW$
Initial velocity	0.0
Pressure difference(bottom-top)	120Pa
Re number	2500
Simulation time	$30\ min$



(a)



(b)

Figure 2. Vertical temperature profiles (in °C) for a tunnel without slope 100 m left and right from the heat source at various times: (a) the low-Mach number model (1), (b) the standard incompressible Navier-Stokes model with Boussinesq approximation.

If we compare the Figure (1a) with the Figure (1b) we see that the temperature fronts are moving with different velocities, i.e. the velocity coming from the Boussinesq approximation model is lower than in the low-Mach number model (1). This fact is clear in the literature where it is claimed that the bouyancies forces are not so strong when simulated with the Boussinesq approximation model. Indeed because the heat transfer towards the tunnel ends is not so fast as in the low-Mach number model, the temparatures in the Boussinesq approximation are higher.

3.2. **Tunnel with slope.** As in the previous example the tunnel configuration data are listed in Table I. The only modification here is that the tunnel has a slope of 3% upwards from the left to the right end. The same features of both simulations are seen also in this case (see Figure 2a,b).

4. Conclusions

Mathematical models which describe fire accidents in tunnels should model low-Mach number flows together with large temperature gradients. In the present paper we compared the low-Mach number model proposed in [3] with the standard Boussinesq approach for fluid flow in the case of two fire examples. As written in Table I we do not use the real Reynolds number, indeed the numerical examples presented here have not to be seen as a comparison with the real experiment data. This is a preliminary step in this direction. The effect of turbulence will be the subject of further investigations.

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References

[1] PIARC 1999, *Fire and smoke control in road tunnels, PIARC Commettre on road tunnels (C5)*, 05.05.B, World Road Association, (1999).

[2] Gasser, I., Struckmeier, J., *An asymptotic-induced one-dimensional model to describe fires in tunnels*, Math. Meth. in Appl. Sci, vol25(14) (2002), pp. 1231-1249.

[3] Gasser, I., Struckmeier, J., Teleaga, I., *Modeling and simulation of tunnel fires*, Int. J. Num. Meth. Fluids, vol. 44(3) (2004), pp. 277-296.

[4] Gasser, I., Struckmeier, J., Teleaga, I., *Modeling and simulation of fires in vehicle tunnels*, PAMM, vol. 3(1) (2003), pp. 400-4001.

[5] Turkel, E., Radespiel, R., Kroll, N., *Assessment of preconditioning methods for multidimensional aerodynamics*, Computer and Fluids, vol. 26(6) (1997), pp 613-634.

[6] Gray, D. D., Giorgini, A., *The validity of the Boussinesq approximation for liquids and gases*, vol. 19 (1976), pp. 545-551.

University of Hamburg, Department of Mathematics, Germany
E-mail address: gasser@math.uni-hamburg.de

University of Hamburg, Department of Mathematics, Germany
E-mail address: struckmeier@math.uni-hamburg.de

TU Darmstadt, Department of Mathematics, Germany
E-mail address: teleaga@mathematik.tu-darmstadt.de