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QUASIPOSITIVE STURM-LIOUVILLE PROBLEM

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Abstract. We explain a new approach for investigation of quasilinear boundary problem by means of Sturm-Liouville problem.

1. **The main result**

In this paper, we consider the following nonlinear problem: find a classical solution $u \in C^2[0,1]$ of equation

$$-u''(x) + p(u(x), u'(x), x) \cdot u(x) = f(x), \tag{1}$$

under Dirichlet condition

$$u(0) = u(1) = 0. \tag{2}$$

We assume that

$$p \in C^0(\mathbf{R} \times \mathbf{R} \times [0,1]), \quad f \in C^0[0,1] \tag{3}$$

We assume that the function p is non-negative:

$$p(u,t,x) \geq 0 \text{ for any } (u,t,x) \in \mathbf{R} \times \mathbf{R} \times [0,1] \tag{4}$$

and there are such constant $C > 0$ and continuous function $c : \mathbf{R} \to \mathbf{R}$ that

$$p(u,t,x) \leq C(c(u) + t^2). \tag{5}$$

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Let u^* be a solution of the *nonlinear* problem (1), (2). Then u^* is the solution of the *linear* problem

$$[-\frac{d^2}{dx^2}+q(x)]u(x)=f(x), \quad u(0)=u(1)=0 \tag{6}$$

with the positive operator $[-d^2/dx^2+q(x)]$ and the non-negative potential

$$q(x)=p(u^*(x),{u^*}'(x),x)\geq 0.$$

Therefore the problem (1), (2) will be named *quasipositive.* Looking at the solution of nonlinear problem (1), (2) as a solution of linear problem (6), we shall introduce a new approach of passage from the boundary problem to a fixed point equation. There are several methods of passage (see [1-3]). Our approach is analogous to D.Gilbarg and N.Trudinger one ([2], chapter 11.3). The eigenfunction theory for quasipositive operators is developed in our papers [4, 5].

We formulate the principal result. Let

$$K(x,\tau)=\begin{cases}(1-x)\tau, \ 0\leq\tau\leq x,\\ x(1-\tau), \ x<\tau\leq 1.\end{cases}$$

be the Green function of boundary problem

$$-u''(x)=f(x), \quad u(0)=u(1)=0.$$

Theorem 1. *The problem (1)-(5) has at least one solution* $u\in C^2[0,1]$*. For any solution the estimate*

$$\int_0^1(u'(x))^2dx\leq\int_0^1\left(\int_0^1 K'_x(x,\tau)f(\tau)d\tau\right)^2dx. \tag{7}$$

is true.

We note that the estimate (7) does not depend on "potential" $p=p(u,t,x)$. It is a direct consequence of the non-negative condition (4).

2. **The operator equation**

Introduce following notations. As usual, we denote $L_k(0,1)$ $(k = 1,2)$ the space of functions on $(0,1)$ which are k integrable. The Sobolev space of functions $u \in L_2(0,1)$ with distributional derivative which are integrable square we denote by $W_2^1(0,1)$; $\overset{\circ}{W_2^1}(0,1) \subset W_2^1(0,1)$ is the closure in $W_2^1(0,1)$ of subspace of C^∞-functions, which are equal to zero outside some segment $[\alpha,\beta] \subset (0,1)$. The norm of $u \in W_2^1(0,1)$ is $||u||_1 = \sqrt{\int_0^1((u')^2(x) + u^2(x))dx}$; the norm of $u \in \overset{\circ}{W_2^1}(0,1)$ is $||u||_1^\circ = \sqrt{\int_0^1 (u')^2(x)dx}$. The norms $||u||_1$ and $||u||_1^\circ$ are equivalent on the space $\overset{\circ}{W_2^2}(0,1)$ due to the boundary condition (2) (see [1], chapter 13.7). Moreover, space $\overset{\circ}{W_2^1}(0,1)$ is Hilbert one with the inner product $(u,v)^\circ = \int_0^1 u'v'dx$.

First we are interested in solutions (of problem (1), (2)) from the space $\overset{\circ}{W_2^1}(0,1)$. As usual, multiplying both sides of the equation (1) by $v \in \overset{\circ}{W_2^1}(0,1)$ and integrating by parts, we get

$$\int_0^1 u'v'dx + \int_0^1 \left(\int_0^1 K(x,\tau)p(u(\tau),u'(\tau),\tau)u(\tau)d\tau\right)' v'dx$$
$$= \int_0^1 \left(\int_0^1 K(x,\tau)f(\tau)\right)' v'dx. \tag{8}$$

A function $u \in \overset{\circ}{W_2^1}(0,1)$ is called a weak solution of the problem (1), (2) if for every $v \in \overset{\circ}{W_2^1}(0,1)$ the equation (8) is valid. By the inner product, the identity (8) is of the following form

$$(u,v)^\circ + (P(u),v)^\circ = (\mathbf{f},v)^\circ, \tag{9}$$

where

$$P : \overset{\circ}{W_2^1}(0,1) \to \overset{\circ}{W_2^1}(0,1), \quad P(u) = \int_0^1 K(x,\tau)p(u(\tau),u'(\tau),\tau)u(\tau)d\tau, \tag{10}$$

$$\mathbf{f} \in \overset{\circ}{W_2^1}(0,1), \quad \mathbf{f} = \int_0^1 K(x,\tau)f(\tau)d\tau. \tag{11}$$

Since (9) is valid for every function $v \in \overset{\circ}{W_2^1}(0,1)$, the identity (9) is equivalent to the operator equation

$$u + P(u) = \mathbf{f}. \tag{12}$$

3. The quasilinear representation of P

Now we investigate the operator P (see (10)) in more detail. By $L(\overset{\circ}{W}{}_2^1(0,1))$ denote the Banach space of continuous linear maps $\mathbf{A}$ which operate in $\overset{\circ}{W}{}_2^1(0,1)$ and by $Lis(\overset{\circ}{W}{}_2^1(0,1)) \subset L(\overset{\circ}{W}{}_2^1(0,1))$ the open subset of linear isomorphisms. As usually, the norm $||\mathbf{A}|| = \sup ||\mathbf{A}v||_1^\circ$ where $||v||_1^\circ = 1$. Consider the map

$$A : \overset{\circ}{W}{}_2^1(0,1) \to L(\overset{\circ}{W}{}_2^1(0,1)), \quad A(u) = \mathbf{A} \ \text{ that } \ \forall v \in \overset{\circ}{W}{}_2^1(0,1)$$

$$\mathbf{A}v = \int_0^1 K(x,\tau) p(u(\tau), u'(\tau), \tau) v(\tau) d\tau,$$

Clearly $P(u) = A(u)u$. We shall call A the quasilinear representation of the map P [5]. Now the equation (12) is of the following form

$$(\mathbf{E} + A(u))u = \mathbf{f}, \tag{13}$$

where $\mathbf{E}$ is identity mapping. Properties of the map A is in next lemma.

Lemma 1. *1) For every $u \in \overset{\circ}{W}{}_2^1(0,1)$ the linear operator $A(u) \in L(\overset{\circ}{W}{}_2^1(0,1))$ is completely continuous.*

2) The map A is completely continuous.

3) For every $u \in \overset{\circ}{W}{}_2^1(0,1)$ the map $\mathbf{E} + A(u) \in Lis(\overset{\circ}{W}{}_2^1(0,1))$ and

$$||(\mathbf{E} + A(u))^{-1}|| < 1. \tag{14}$$

Proof. Since for any $u, v \in \overset{\circ}{W}{}_2^1(0,1)$

$$((A(u)v)(x))' = \int_0^1 K'_x(x,\tau) p(u(\tau), u'(\tau), \tau) v(\tau) dt$$

and the function $r(\xi) = p(u(\xi), u'(\xi), \xi)v(\xi) \in L_1(0,1)$ is integrable one (see (5)), the function $(A(u)v)' \in C^0[0,1]$. Thus the map $\mathbf{A} = A(u) : \overset{\circ}{W}{}_2^1(0,1) \to \{C^1[0,1] \cap (2)\}$ is continuous. Embedding $im : \{C^1[0,1] \cap (2)\} \subset \overset{\circ}{W}{}_2^1(0,1)$ is completely continuous ([1], chapter 26.24). Hence the linear operator $A(u) = im \cdot A(u)$ is completely continuous as the composition of continuous and completely continuous maps [6].

To prove the second statement, we represent the map A in the form

$$A(u)v = -x \int_0^1 \left(\int_0^\tau \left(\int_0^\xi p(u(\nu), u'(\nu), \nu) d\nu \right) v'(\xi) d\xi \right) d\tau +$$

$$\int_0^x \left(\int_0^\tau \left(\int_0^\xi p(u(\nu), u'(\nu), \nu) d\nu \right) v'(\xi) d\xi \right) d\tau.$$

Write map A as the composition of four maps: $A = \delta \cdot \gamma \cdot \beta \cdot \alpha$, where

$$\alpha : \overset{\circ}{W}{}_2^1 (0,1) \to L_1(0,1), \quad \alpha(u) := p(u(\xi), u'(\xi), \xi) = q$$

$$\beta : L_1(0,1) \to C^0[0,1], \quad \beta(q) := \int_0^\xi q(\nu) d\nu = s;$$

$$\gamma : C^0[0,1] \subset L_2(0,1), \quad \gamma(s) := s \text{ is the natural embedding;}$$

$$\delta : L_2(0,1) \to L(\overset{\circ}{W}{}_2^1 (0,1)), \quad \delta(s) = \mathbf{A} \text{ that } \forall v \in \overset{\circ}{W}{}_2^1 (0,1)$$

$$\mathbf{A}(v) = -x \int_0^1 \left(\int_0^\tau s(\xi) v'(\xi) d\xi \right) d\tau + \int_0^x \left(\int_0^\tau s(\xi) v'(\xi) d\xi \right) d\tau.$$

These maps are continuous and the map γ is completely continuous. This completes the proof of the second statement.

For any $u, v \in \overset{\circ}{W}{}_2^1 (0,1)$ we have

$$(||(\mathbf{E} + A(u))v||_1^\circ)^2 = (v,v)^\circ + 2(A(u)v, v)^\circ + (A(u)v, A(u)v)^\circ \geq$$

$$(||v||_1^\circ)^2 + 2(A(u)v, v)^\circ.$$

Since (see (4))

$$(A(u)v, v)^\circ = \int_0^1 p(u(x), u'(x), x) v^2(x) dx \geq 0,$$

then

$$(||(\mathbf{E} + A(u))v||_1^\circ)^2 \geq (||v||_1^\circ)^2.$$

Whence we obtain the third statement. □

4. **Proof of Theorem**

Next step is the passage from the non-homogeneous equation (13) to a fixed point equation.

Lemma 2. *1) The equation (13) is equivalent to the operator equation*

$$u = (\mathbf{E} + A(u))^{-1}\mathbf{f}. \tag{15}$$

2) For any weak solution u the following a priori estimate is valid:

$$||u||_1^\circ \leq ||\mathbf{f}||_1^\circ. \tag{16}$$

3) The map

$$B : \overset{\circ}{W}{}_2^1 (0,1) \to \overset{\circ}{W}{}_2^1 (0,1), \quad B(u) := (\mathbf{E} + A(u))^{-1}\mathbf{f}$$

is completely continuous.

Proof. The first statement follows from the third statement of Lemma 1. The second statement follows from the first one and third statement of Lemma 1 (see (14)).

The map B is the composition:

$$u \to A(u) \to \mathbf{E} + A(u) \to (\mathbf{E} + A(u))^{-1} \to (\mathbf{E} + A(u))^{-1}\mathbf{f}.$$

The first map is completely continuous (see the second statement of Lemma 1) and the others maps are continuous. This completes the proof [6]. □

Note that the map $B = (\mathbf{E} + A(u))^{-1}\mathbf{f}$ depends on the u in the operator part only. Thus properties of equation (15) follow from properties of map A.

To proof Theorem, we apply Leray-Schauder degree. Let the ball $T_R = \{u \in \overset{\circ}{W}{}_2^1 (0,1) : ||u||_1^\circ \leq R\}$, where the constant $R > ||\mathbf{f}||_1^\circ$. Let the sphere $S_R = \{u \in \overset{\circ}{W}{}_2^1 (0,1) : ||u||_1^\circ = R\}$. By (14) and (15), for any $u \in S_R$ we obtain $||u||_1^\circ > ||B(u)||_1^\circ$. Therefore on S_R the completely continuous vector field $u - B(u) \neq 0$ and degree of B is equal to one [6]. Consequently there is a solution $u \in T_R$ of equation (15). The existence of a weak solution is proved.

By (16) and (11) we obtain

$$\int_0^1 (u'(x))^2 dx \le \int_0^1 \left\{ \left(\int_0^1 K(x,\tau) f(\tau) d\tau \right)'_x \right\}^2 dx = \int_0^1 \left(\int_0^1 K'_x(x,\tau) f(\tau) d\tau \right)^2 dx.$$

The estimate (7) is proved.

Actually, the weak solution $u \in \overset{\circ}{W}{}_2^1 (0,1)$ is the classical solution, i.e. $u \in C^2[0,1]$. This follows from well known theorem about regularity of weak solution (see [1], §17). Theorem is proved. □

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