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UNIQUENESS ALGEBRAIC CONDITIONS IN THE STUDY OF SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. The purpose of this paper is to give some algebraic conditions for the coefficients of an second order differential system in order to obtain some uniqueness and comparison results.

1. Introduction

Let us consider the following second order differential system:

$$Lu := \delta^2 I_n \frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0, \delta > 0, \tag{1}$$

where $B, C \in C([a,b], M_n(\mathbb{R}))$, and the following statement:

$$\left.\begin{array}{c} u \in C^2([a,b], \mathbb{R}^n) \\ Lu = 0,\ in\ ]a,b[\\ u(a) = u(b) = 0 \end{array}\right\} \Longrightarrow u \equiv 0\, in\ [a,b] \tag{2}$$

It is well known the fact that if u satisfies a maximum principle, then the statement (2) automatically take place. The aim of this paper is to determine effective algebraic conditions for B, C such that the statement (2) to take place, without using a maximum principle. Let $A \in M_n(\mathbb{R})$, J the Jordan normal form of A. We know that there exist a nonsingular matrix T such that $A = TJT^{-1}$.

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We will denote:

$$\begin{aligned}
\widetilde{\alpha} &= \begin{cases} \frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, \lambda_k \in \mathbb{R} \\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k \; Re\lambda_k, \lambda_k \in \mathbb{C}\backslash\mathbb{R} \end{cases} \\
\gamma_F &= \|T\|_F \cdot \left\|T^{-1}\right\|_F \\
m_F &= \|J - \widetilde{\alpha} I\|_F
\end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of a matrix (see [2]).

We shall use the following result given in [2]:

Theorem 1. *Let* $\varphi_{\|\cdot\|} : \mathbb{R} \to \mathbb{R}, \varphi_{\|\cdot\|}(\alpha) = \|A - \alpha I_n\|$ *,* $\|\cdot\|$ *being one of the following norms:* $\|\cdot\|_F$*,* $\|\cdot\|_1$*,* $\|\cdot\|_2$*,* $\|\cdot\|_\infty$*. In these conditions:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \sqrt{n}\gamma_F m_F.$$

Remark 1. *In case of euclidean norm* $\|\cdot\|_F$ *and spectral norm* $\|\cdot\|_2$ *we have that* $\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \gamma_F m_F$ *(see [2]). Because* $n \geq 2$*, if* $m_F \neq 0$*, then:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) < \sqrt{n}\gamma_F m_F.$$

Conditions determined here will be very useful to obtain comparison results (see Section 3 of the paper).

2. Establishing the conditions in which the statement (2) take place

Let $u \in C^2([a,b], \mathbb{R}^n)$, $u \neq 0$, a solution of the system (1) with the property that $u(a) = u(b) = 0$. We have:

$$u^* Lu = \delta^2 u^* \frac{d^2u}{dx^2} + u^* B(x) \frac{du}{dx} + u^* C(x) u$$

$$\delta^2 \frac{d}{dx}\left(u^* \frac{du}{dx}\right) = u^* Lu - u^* B(x) \frac{du}{dx} - u^* C(x) u + \delta^2 \frac{du^*}{dx}\frac{du}{dx}.$$

If we integrate on $[a,b]$, we obtain:

$$\int_a^b \left(\delta^2 \frac{du^*}{dx}\frac{du}{dx} - u^* B(x)\frac{du}{dx} - u^* C(x)u\right)dx = 0.$$

Let us denote

$$E := \delta^2 \frac{du^*}{dx}\frac{du}{dx} - u^* B(x)\frac{du}{dx} - u^* C(x)u.$$

We shall show that under some assumptions for the coefficients B and C this expression is positive which will imply the fact that the integral can not be zero on $[a,b]$, only if $E \equiv 0$. Let $u = R \cdot e$, where $R = \|u\| = \left(\sum\limits_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}$,

$$e \in C^2([a,b],\mathbb{R}^n),\ e = \begin{pmatrix} e_1 \\ . \\ . \\ . \\ e_n \end{pmatrix},\ e^* = (e_1,...,e_n),\ \|e\| = \left(\sum_{i=1}^{n} e_i^2\right)^{\frac{1}{2}} = 1.$$

A simple computation shows us that

$$E = \delta^2 (R')^2 + e^* B(x) eRR' - (e^* Le) R^2 \tag{3}$$

where

$$e^* Le = -\delta^2 \|e'\|^2 + e^* B(x) e' + e^* C(x) e.$$

The quadric form (3) is positive if and only if

$$[e^* B(x) e]^2 + 4\delta^2 e^* Le \le 0. \tag{4}$$

It is simple to see that:

$$e^* Le \le \frac{1}{4\delta^2}\left\|B(x) - \widetilde{\beta}(x) I_n\right\|^2 + e^* C(x) e$$

From Theorem 1 we know that for every $x \in\,]a,b[$ there exist $\widetilde{\beta}(x) \in \mathbb{R}$ such that

$$\left\|B(x) - \widetilde{\beta}(x) I_n\right\| \le \gamma_F m_F.$$

If we suppose that $m_F \ne 0$, than we have:

$$\left\|B(x) - \widetilde{\beta}(x) I_n\right\| < \gamma_F m_F.$$

Under assumption that

$$e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}n\left(\gamma_{F}m_{F}\right)^{2},\forall x\in\left]a,b\right[\tag{5}$$

we obtain:

$$e^{*}Le\leq\frac{1}{4\delta^{2}}\left[\left\|B\left(x\right)-\widetilde{\beta}\left(x\right)I_{n}\right\|^{2}-n\left(\gamma_{F}m_{F}\right)^{2}\right]:=-p^{2}\left(x\right)<0. \tag{6}$$

Supposing that

$$e^{*}B\left(x\right)e\leq2\delta p\left(x\right), \tag{7}$$

we observe that the relation (4) take place. In conclusion if (5) and (7) take place, then the quadric form (3) is positive and that means that the integral can not be identically null, only if $E\equiv0$. But, if $E\equiv0$, then

$$\frac{d}{dx}\left(u^{*}\frac{du}{dx}\right)=0,$$

meaning that $\|u\|^{2}$ is constant. Because $u\left(a\right)=u\left(b\right)=0$, we obtain that $u\equiv0$. In this way, if $m_{F}\neq0$, we obtain the following result:

Theorem 2. *Suppose that:*

1. $e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}n\left(\gamma_{F}m_{F}\right)^{2},\forall x\in\left]a,b\right[;$
2. $e^{*}B\left(x\right)e\leq2\delta p\left(x\right);$

$\forall e\in C^{2}\left(\left[a,b\right],\mathbb{R}^{n}\right),\|e\|=\left(\sum\limits_{i=1}^{n}e_{i}^{2}\right)^{\frac{1}{2}}=1$, *with p as in* (6). *In these conditions the statement* (2) *take place.*

Example 1. *If we consider system* (1), *in the case $n=2$, with* $B=\begin{pmatrix}a_{1}&a_{2}\\a_{3}&a_{1}\end{pmatrix}$, $a_{2},a_{3}>0$, $a_{1}^{2}\leq a_{2}^{2}+4a_{2}a_{3}+a_{3}^{2}$, *we have an example of matrix B which verifies* (7), *with* $p^{2}=\frac{1}{4\delta^{2}}\left(3a_{2}^{2}+8a_{2}a_{3}+3a_{3}^{2}\right).$

Remark 2. *If $m_{F}=0$, then the statement* (2) *still take place if*

$$e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}\left(\widetilde{\beta}\left(x\right)\right)^{2},\forall x\in\left]a,b\right[.$$

3. A comparison result

Let us consider the following second order differential systems:

$$Lu := \lambda^2 \frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0 \tag{8}$$

$$Mv := \mu^2 \frac{d^2u}{dx^2} + Q(x)v = 0, \tag{9}$$

with $B \in C^1([a,b], M_n(\mathbb{R})), C, Q \in C([a,b], M_n(\mathbb{R})), \lambda > \mu > 0$. Using the same method as in section 2 of this paper we obtain the following result (in case $m_F \neq 0$):

Theorem 3. *Suppose:*

1. *Q is symmetric;*
2. *There exist a solution matrix S of system* (9) *such that* $\det S(x) \neq 0$ *in* $[a,b]$ *and the matrix* $\frac{dS}{dx}S^{-1}$ *is symmetric..*

If:

(i): $e^*(C(x) - Q(x))e \leq -\frac{1}{4\delta^2} n(\gamma_F m_F)^2, \forall x \in]a,b[;$

(ii): $e^*B(x)e \leq 2\delta p(x), \forall x \in]a,b[.$

$\forall e \in C^2([a,b], \mathbb{R}^n), \|e\| = \left(\sum\limits_{i=1}^{n} e_i^2\right)^{\frac{1}{2}} = 1,$ *with* $\delta^2 = \lambda^2 - \mu^2$, $p^2(x) = \frac{1}{4\delta^2}\left[n(\gamma_F m_F)^2 - \left\|B(x) - \widetilde{\beta}(x)I_n\right\|^2\right] > 0$, *then the system* (8) *is non-oscillatory.*

Remark 3. *If $m_F = 0$, conditions* (*i*) *and* (*ii*) *from Theorem 3 are reduced to:*

$$e^*(C(x) - Q(x))e \leq -\frac{1}{4\delta^2}\left(\widetilde{\beta}(x)\right)^2, \forall x \in]a,b[.$$

Remark 4. *Theorem 3 improve a result given by I.A. Rus in* [5].

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