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ON REVERSE HILBERT TYPE INEQUALITIES

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Abstract. In this paper we establish a new inverse inequality of Hilbert type for a finite number of positive sequences of real numbers. The integral analogue of the inequality are also proved. The results of this paper reduce to that of B. G. Pachpatte.

1. Introduction

In recent years several authors(see [1], [2], [3], [4], [5], [6], [7], [8]) have given considerable attention to Hilbert's inequalities and Hilbert type inequalities and their various generalizations. In particular, in 1988, B. G. Pachpatte[1] proved two new inequalities similar to Hilbert's inequality[9, P.226]. These two new results can be stated as follows, respectively:

Theorem A. *Let $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ with $a_0 = b_0 = 0$ and let $\{p_m\}$ and $\{q_n\}$ be two positive sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k, r are natural numbers and define $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be two real-valued nonnegative, convex and submultiplicative functions defined on $R_+ = [0, \infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} \leq M(k,r)\bigg(\sum_{m=1}^{k}\Big(k-m+1\Big)\Big(p_m\phi(\frac{\nabla(a_m)}{p_m})\Big)^2\bigg)^{1/2}$$

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$$\times\Bigg(\sum_{n=1}^{r}\Big(r-n+1\Big)\Big(q_n\psi\big(\frac{\nabla(b_n)}{q_n}\big)\Big)^2\Bigg)^{1/2}, \tag{1}$$

where

$$M(k,r)=\frac{1}{2}\Bigg(\sum_{m=1}^{k}\Big(\frac{\phi\big(P_m\big)}{P_m}\Big)^2\Bigg)^{1/2}\Bigg(\sum_{n=1}^{r}\Big(\frac{\psi\big(Q_n\big)}{Q_n}\Big)^2\Bigg)^{1/2},$$

and $\nabla(a_m)=a_m-a_{m-1}, \nabla(b_n)=b_n-b_{n-1}$.

Theorem B. *Let* $f\in C^1[[0,x),R_+], g\in C^1[[0,y),R_+]$ *with* $f(0)=g(0)=0$ *and let* $p(\sigma)$ *and* $q(\tau)$ *be two positive functions defined for* $\sigma\in[0,x)$ *and* $\tau\in[0,y)$, *and* $P(s)=\int_0^s p(\sigma)d\sigma$ *and* $Q(t)=\int_0^t q(\tau)d\tau$ *for* $s\in[0,x)$ *and* $t\in[0,y)$, *where* x,y *are positive real numbers. Let* ϕ *and* ψ *be as in Theorem A. Then*

$$\int_0^x\int_0^y\frac{\phi\Big(f(s)\Big)\psi\Big(g(t)\Big)}{s+t}dsdt\le L(x,y)\Bigg(\int_0^x(x-s)\Big(p(s)\phi\Big(\frac{f'(s)}{p(s)}\Big)\Big)^2ds\Bigg)^{1/2}$$

$$\times\Bigg(\int_0^y(y-t)\Big(q(t)\psi\Big(\frac{g'(t)}{q(t)}\Big)\Big)^2dt\Bigg)^{1/2}, \tag{2}$$

where

$$L(x,y)=\frac{1}{2}\Bigg(\int_0^x\Bigg(\frac{\phi\Big(P(s)\Big)}{P(s)}\Bigg)^2ds\Bigg)^{1/2}\Bigg(\int_0^y\Bigg(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\Bigg)^2dt\Bigg)^{1/2},$$

and $'$ *denotes the derivative of a function.*

The main purpose of this paper is to establish reverse forms of the above two inequalities.

2. Main results

Theorem 1. *Let* $\{a_{i,m_i}\}(i=1,2,\dots,n)$ *be* n *positive sequences of real numbers defined for* $m_i=1,2,\dots,k_i$ *with* $a_{i,0}=0(i=1,2,\dots,n)$, *where* $k_i(i=1,\dots,n)$ *are the natural numbers. Let* $\{p_{i,m_i}\}$ *be* n *positive sequences of real numbers defined for* $m_i=1,2,\dots,k_i(i=1,2,\dots,n)$. *Set* $P_{i,m_i}=\sum_{s_i=1}^{m_i}p_{i,s_i}(i=1,2,\dots,n)$. *Let* $\phi_i(i=1,2,\dots,n)$ *be* n *real-valued nonnegative concave, supermultiplicative and non-decreasing functions defined on* $R_+=[0,+\infty)$. *Let* $\frac{1}{\alpha_i}+\frac{1}{\beta_i}=1, 0<\beta_i<1$

and $\sum_{i=1}^{n}\frac{1}{\alpha_i}=\frac{1}{\alpha}$. *Sut* $A_{i,m_i}^{(p_i)}=\nabla(a_{i,m_i})\cdot a_{i,m_i}^{p_i-1}$, *where the operator* ∇ *is defined by* $\nabla(a_{i,m_i})=a_{i,m_i}-a_{i,m_i-1}(i=1,2,\ldots,n)$ *and* $0\le p_i\le 1$ *are real numbers. Then*

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(\frac{a_{i,m_i}^{p_i}}{p_i})}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}}$$

$$\ge M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}, \quad (3)$$

where

$$M(k_1,k_2,\ldots,k_n)=\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\alpha_i}\right)^{1/\alpha_i}.$$

Proof. By using the following inequality(see Hardy *et al.* [9,P.39])

$$h_i x_{i,m_i}^{h_i-1}(x_{i,m_i}-y_{i,m_i})\le x_{i,m_i}^{h_i}-y_{i,m_i}^{h_i}\le h_i y_{i,m_i}^{h_i-1}(x_{i,m_i}-y_{i,m_i}),$$

where $x_{i,m_i}>0$ and $y_{i,m_i}>0$ and $0\le h_i\le 1(i=1,2,\ldots,n)$, we obtain that

$$a_{i,m_i+1}^{p_i}-a_{i,m_i}^{p_i}\ge p_i(a_{i,m_i+1})^{p_i-1}(a_{i,m_i+1}-a_{i.m_i})=p_i(a_{i,m_i+1})^{p_i-1}\cdot\nabla(a_{i,m_i+1}).$$

Consequently

$$\sum_{m_i=0}^{k_i-1}a_{i,m_i+1}^{p_i}-a_{i,m_i}^{p_i}=a_{i,k_i}^{p_i}\ge p_i\sum_{m_i=0}^{k_i-1}\nabla(a_{i,m_i+1})\cdot a_{i,m_i+1}^{p_i-1}=p_i\sum_{m_i=1}^{k_i}A_{i,m_i}^{(p_i)}.$$

Hence

$$\frac{a_{i,m_i}^{p_i}}{p_i}\ge\sum_{s_i=1}^{m_i}A_{i,s_i}^{(p_i)}. \quad (4)$$

On the other hand, from the following theorem of the Arithmetic and Geometric means[9,p.17]

$$\prod_{i=1}^{n}b_i^{q_i}\le\left(\frac{\sum_{i=1}^{n}q_ib_i}{\sum_{i=1}^{n}q_i}\right)^{\sum_{i=1}^{n}q_i},$$

where $q_i>0,b_i>0$, we easy get the following result

$$\prod_{i=1}^{n}m_i^{1/\alpha_i}\ge\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}, \quad (5)$$

where $\alpha_i<0$.

From (4), (5) and in view of Jensen's inequality and inverse Hölder's inequality[10], we obtain that

$$\prod_{i=1}^{n}\phi_i\left(\frac{a_{i,m_i}^{p_i}}{p_i}\right) \geq \prod_{i=1}^{n}\phi_i\left(\frac{P_{i,m_i}\sum_{s_i=1}^{m_i}p_{i,s_i}\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\phi_i(P_{i,m_i})\cdot\phi_i\left(\frac{\sum_{s_i=1}^{m_i}p_{i,s_i}\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\sum_{s_i=1}^{m_i}p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}m_i^{1/\alpha_i}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$\geq \left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}\prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}. \quad (6)$$

Dividing both sides of (6) by $\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}$ and then taking the sum over m_i from 1 to $k_i (i = 1, 2, \ldots, n)$ and in view of inverse Hölder's inequality, we have

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i\left(\frac{a_{i,m_i}^{p_i}}{p_i}\right)}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}}$$

$$\geq \prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}\right)$$

$$\geq \prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\alpha_i}\right)^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$= M(k_1, k_2, \ldots, k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$= M(k_1, k_2, \ldots, k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i - m_i + 1)\left(p_{i,m_i}\phi_i\left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}.$$

The proof is complete.

Remark 1. Taking for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$ in (3), (3) changes to

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} \phi_i(\frac{a_{i,m_i}^{p_i}}{p_i})}{(m_1 + \cdots + m_n)^{-n/(n-1)}}$$

$$\geq \bar{M}(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left(p_{i,m_i} \phi_i \left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}} \right) \right)^{(n-1)/n} \right)^{n/(n-1)}, \tag{7}$$

where

$$\bar{M}(k_1, k_2, \ldots, k_n) = n^{n/(n-1)} \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} \left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right)^{-(n-1)} \right)^{-1/(n-1)}.$$

Taking for $n = 2$ and $p_i = 1(i = 1, 2)$ in (7), (7) becomes

$$\sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{\phi_1(a_{1,m_1})\phi_2(a_{2,m_2})}{(m_1 + m_2)^{-2}} \geq$$

$$\geq M(k_1, k_2)\Big(\sum_{m_1=1}^{k_1} \Big(k_1 - m_1 + 1\Big)\Big(p_{1,m_1}\phi_1\big(\frac{\nabla(a_{1,m_1})}{p_{1,m_1}}\big)\Big)^{1/2}\Big)^2$$

$$\times\Big(\sum_{m_2=1}^{k_2} \Big(k_2 - m_2 + 1\Big)\Big(p_{2,m_2}\phi_2\big(\frac{\nabla(a_{2,m_2})}{p_{2,m_2}}\big)\Big)^{1/2}\Big)^2, \tag{8}$$

where

$$M(k_1, k_2) = 4 \left(\sum_{m_1=1}^{k_1} \Big(\frac{\phi_1\big(P_{1,m_1}\big)}{P_{1,m_1}} \Big)^{-1} \right)^{-1} \left(\sum_{m_2=1}^{k_2} \Big(\frac{\phi_2\big(P_{2,m_2}\big)}{P_{2,m_2}} \Big)^{-1} \right)^{-1},$$

and

$$\nabla(a_{1,m_1}) = a_{1,m_1} - a_{1,m_1-1}, \nabla(a_{2,m_2}) = a_{2,m_2} - a_{2,m_2-1}.$$

Inequality (8) is just an reverse form of inequality (1) which was stated in the introduction.

Theorem 2. *Let* $f_i(\sigma_i) \in C^1[[0, x_i), [0, \infty)], i = 1, \ldots, n,$ *with* $f_i(0) = 0$, *Let* $p_i(\sigma_i)$ *be* n *positive functions defined for* $\sigma_i \in [0, x_i)(i = 1, 2, \cdots, n)$ *and define* $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$, for $s_i \in [0, x_i)$, *where* x_i *are positive real numbers and set* $F_{i,\sigma_i}^{(p_i)} = f_i'(\sigma_i) f_i^{p_i-1}(\sigma_i)$, *where* p_i *are real numbers. Let* $\phi_i(i = 1, 2, \ldots, n)$ *be* n

real-valued nonnegative concave and supermultiplicative functions defined on $R_+ = [0, +\infty)$. *Let* α_i, β_i *and* α *be as in Theorem 1. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i\left(\frac{f_i^{p_i}(s_i)}{p_i}\right)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n$$

$$\geq L(x_1, \cdots, x_n) \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i) \phi_i\left(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)}\right) \right)^{\beta_i} ds_i \right)^{1/\beta_i}, \tag{9}$$

where

$$L(x_1, \cdots, x_n) = \prod_{i=1}^n \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{\alpha_i} ds_i \right)^{1/\alpha_i}.$$

Proof. From the hypotheses, we have

$$f_i^{p_i}(s_i) = p_i \int_0^{s_i} F_{i,\sigma_i}^{(p_i)} d\sigma_i, s_i \in [0, x_i).$$

By using Jensen integral inequality and inverse Hölder integral inequality and notice that $\phi_i (i = 1, 2, \cdots, n)$ are n real-valued supermultiplicative functions, it is easy to observe that

$$\prod_{i=1}^n \phi_i \left(\frac{f_i^{p_i}(s_i)}{p_i} \right) = \prod_{i=1}^n \phi_i \left(\frac{P_i(s_i) \int_0^{s_i} p_i(\sigma_i) \frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} d\sigma_i}{\int_0^{s_i} p_i(\sigma_i) d\sigma_i} \right)$$

$$\geq \prod_{i=1}^n \phi_i(P_i(s_i)) \phi_i \left(\frac{\int_0^{s_i} p_i(\sigma_i) \frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} d\sigma_i}{\int_0^{s_i} p_i(\sigma_i) d\sigma_i} \right) \geq \prod_{i=1}^n \frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \int_0^{s_i} p_i(\sigma_i) \phi_i \left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} \right) d\sigma_i$$

$$\geq \prod_{i=1}^n \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right) s_i^{1/\alpha_i} \left(\int_0^{s_i} \left(p_i(\sigma_i) \phi_i \left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} \right) \right)^{\beta_i} d\sigma_i \right)^{1/\beta_i}. \tag{10}$$

In view of inequality (5) and integrating two sides of (10) over s_i from 0 to $x_i (i = 1, 2, \ldots, n)$ and noticing reverse Hölder integral inequality, we observe that

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(\frac{f_i^{p_i}(s_i)}{p_i})}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n$$

$$\geq \prod_{i=1}^{n} \int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right) \left(\int_0^{s_i} \left(p_i(\sigma_i)\phi_i\left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i$$

$$\geq \prod_{i=1}^{n} \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{\alpha_i} ds_i \right)^{1/\alpha_i} \left(\int_0^{x_i}\int_0^{s_i} \left(p_i(\sigma_i)\phi_i\left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i ds_i \right)^{1/\beta_i}$$

$$= L(x_1, \ldots, x_n) \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)})\right)^{\beta_i} ds_i \right)^{1/\beta_i}.$$

This completes the proof of Theorem 2.

Remark 2. Taking for $\beta_i = \frac{n-1}{n} (i = 1, \ldots, n)$ in (9), (9) changes to

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(\frac{f_i^{p_i}(s_i)}{p_i})}{(s_1 + \cdots + s_n)^{-n/(n-1)}} ds_1 \cdots ds_n$$

$$\geq \bar{L}(x_1, \cdots, x_n) \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)})\right)^{(n-1)/n} ds_i \right)^{n/(n-1)}, \quad (11)$$

where

$$\bar{L}(x_1, \cdots, x_n) = n^{n/(n-1)} \prod_{i=1}^{n} \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{-(n-1)} ds_i \right)^{-1/(n-1)}.$$

Taking $n = 2$ and $p_i = 1$ to (11), (11) changes to

$$\int_0^{x_1}\int_0^{x_2} \frac{\phi_1\Big(f_1(s_1)\Big)\phi_2\Big(f_2(s_2)\Big)}{(s_1+s_2)^{-2}} ds_1 ds_2 \geq$$

$$\geq L(x_1, x_2)\Bigg(\int_0^{x_1} (x_1 - s_1)\Big(p_1(s_1)\phi_1\Big(\frac{f_1'(s_1)}{p_1(s_1)}\Big)\Big)^{1/2} ds_1 \Bigg)^2$$

$$\times\Bigg(\int_0^{x_2} (x_2 - s_2)\Big(p_2(s_2)\phi_2\Big(\frac{f_2'(s_2)}{p_2(s_2)}\Big)\Big)^{1/2} ds_2 \Bigg)^2, \quad (12)$$

where

$$\bar{L}(x_1, x_2) = 4\Bigg(\int_0^{x_1} \Big(\frac{\phi_1\Big(P_1(s_1)\Big)}{P_1(s_1)}\Big)^{-1} ds_1 \Bigg)^{-1} \Bigg(\int_0^{x_2} \Big(\frac{\phi_2\Big(P_2(s_2)\Big)}{P_2(s_2)}\Big)^{-1} ds_2 \Bigg)^{-1}.$$

Inequality (12) is just an reverse form of inequality (2) which was stated in the introduction.

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