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ON THE BASIC PROPERTIES OF DISCONTINUOUS FLOWS

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Abstract. In this paper, we define discontinuous dynamical systems which can be used as models of various processes in mechanics, electronics, biology and medicine. We find sufficient conditions to guarantee the existence of such systems. These conditions are easy to verify.

1. **Introduction and preliminaries**

A book [1] edited by D.V. Anosov and V.I. Arnold considers two fundamentally different Dynamical Systems (DSs) : flows and cascades. Roughly speaking, flows are DSs with continuous time and cascades are DSs with discrete time. One of the most important theoretical problem is to consider *Discontinuous Dynamical Systems* ($DDSs$). That is systems whose trajectories are piecewise continuous curves. It is well-recognized (for example, see [2]) that the general notion of such systems was introduced by Th. Pavlidis [3], although particular examples (the mathematical model of clock [4]-[6] and so on) had been discussed before. Some basic elements of the theory are given in [7]-[11]. Analysing the behavior of the trajectories we can conclude that $DDSs$ combine features of vector fields and maps, they can not be reduced to flows or cascades, but are close to flows since time is continuous. That is why we propose to call them also *Discontinuous Flows* (DFs). Applications of $DDSs$ in mechanics, electronics, biology and medicine were considered in [3], [12] - [15]. Chaotic behavior of discontinuous processes was investigated in [13, 16]. One must emphasize that DFs are not *differential equations with discontinuous right side* which often have

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been accepted as $DDSs$ [17]. However, theoretical problems of nonsmooth dynamics and discontinuous maps [18]-[25] are also very close to the subject of our paper. One should also agree that *nonautonomous impulsive differential equations*, which were thoroughly described in [8] and [11], are not DFs.

The paper embodies results that provide sufficient conditions for the existence of a *differentiable DF*. Since DFs have specific smoothness of solutions we call these systems *B-differentiable DFs.* Apparently, it is the first time when notions of $B-$ continuous and $B-$ differentiable dependence of solutions on initial values [27] are applied to described $DDSs$ and sufficient conditions for the continuation of solutions and the group property are obtained. A central auxiliary result of the paper is the construction of a new form of the general autonomous impulsive equation (system (1)). Effective methods of investigation of systems with variable time of impulsive actions were considered in [8, 11], [27]- [31].

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $|| \cdot ||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider a set of strictly ordered real numbers $\{\theta_i\}$, where the set $\mathcal{A}$ of indices is an interval of $\mathbb{Z}/\{0\}$.

Definition 1.1. *The set* $\{\theta_i\}$ *is said to be a sequence of* $\beta - type$ *if the product* $i\theta_i, \quad i \geq 0$ *for all* i *and one of the following alternative cases holds:*

(a) $\{\theta_i\} = \emptyset$;

(b) $\{\theta_i\}$ *is a finite and nonempty set;*

(c) $\{\theta_i\}$ *is an infinite set such that* $|\theta_i| \to \infty$ *as* $|i| \to \infty$.

From the definition, it follows immediately that a sequence of $\beta - type$ does not have a finite accumulation point in $\mathbb{R}$.

Definition 1.2. *A function* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n$ *is said to be from a space* $\mathcal{P}C(\mathbb{R})$ *if*

1. $\varphi(t)$ *is left continuous on* $\mathbb{R}$;
2. *there exists a sequence* $\{\theta_i\}$ *of* $\beta-$ *type such that* φ *is continuous if* $t \neq \theta_i$ *and* φ *has discontinuities of the first kind at the points* θ_i.

Particularly, $C(\mathbb{R}) \subset \mathcal{P}C(\mathbb{R})$.

Definition 1.3. *A function* $\varphi(t)$ *is said to be from a space* $\mathcal{P}C^1(\mathbb{R})$ *if* $\varphi' \in \mathcal{P}C(\mathbb{R})$.

Let T be an interval in $\mathbb{R}$.

Definition 1.4. *We denote by $\mathcal{PC}(T)$ and $\mathcal{PC}^1(T)$ the sets of restrictions of all functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}^1(\mathbb{R})$ on T respectively.*

Let G be an open subset of $\mathbb{R}^n$, G_r be an $r-$ neighbourhood of G in $\mathbb{R}^n$ for a fixed $r > 0$ and $\hat{G} \subset G_r$ be an open subset of $\mathbb{R}^n$. Denote as $\Phi : \hat{G} \longrightarrow \mathbb{R}$ be a function from $C^1(\hat{G})$ and assume that a surface $\Gamma = \Phi^{-1}(0)$ is a subset of $\bar{G}$, where $\bar{G}$ denotes the closure of the set G in $\mathbb{R}^n$. Moreover, define a function $J : \Gamma_r \to \bar{G}$, where Γ_r is an $r-$ neighbourhood of Γ. We shall need the following assumptions.

C1) $\nabla\Phi(x) \neq 0$, $\forall x \in \Gamma$;
C2) $J \in C^1(\Gamma_r), \det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma$.

One can see that the restriction $J|_\Gamma$ is a one-to-one function. Let also $\tilde{\Gamma} = J(\Gamma)$, $\tilde{\Gamma} \subset \bar{G}$. If $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$, $x \in \tilde{\Gamma}$ then $\tilde{\Gamma} = \left\{x \in G |\, \tilde{\Phi}(x) = 0\right\}$. It is easy to verify that $\nabla\tilde{\Phi}(x) \neq 0$, $\forall x \in \tilde{\Gamma}$.

Consider the following impulsive differential equation in the domain $D = \left[G \cup \Gamma \cup \tilde{\Gamma}\right] \setminus \left[(\bar{\Gamma}\backslash\Gamma) \cup \left(\bar{\tilde{\Gamma}}\backslash\tilde{\Gamma}\right)\right]$

$$
\begin{aligned}
&x'(t) = f(x(t)), \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\},\\
&x(t+)|_{x(t-)\in\Gamma\wedge t\geq 0} = J(x(t-)),\\
&x(t-)|_{x(t+)\in\tilde{\Gamma}\wedge t\leq 0} = J^{-1}(x(t+)).
\end{aligned}
\tag{1}
$$

We make the following assumptions which will be needed throughout the paper.

C3) $f \in C^1(G_r)$.
C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$, $\Gamma \cap \left(\bar{\tilde{\Gamma}}\backslash\tilde{\Gamma}\right) = \emptyset$, $\left(\bar{\Gamma}\backslash\Gamma\right) \cap \tilde{\Gamma} = \emptyset$.
C5) $\langle \nabla\Phi(x), f(x)\rangle \neq 0$ if $x \in \Gamma$.
C6) $\left\langle \nabla\tilde{\Phi}(x), f(x)\right\rangle \neq 0$ if $x \in \tilde{\Gamma}$.

2. **Main results**

Definition 2.1. *A function $x(t) \in \mathcal{P}C^1(T)$ with a set of discontinuity points $\{\theta_i\} \subset T$ is said to be a solution of (1) on the interval $T \subset \mathbb{R}$ if it satisfies the following conditions:*

(i) *equation (1) is satisfied at each point $t \in T \backslash \{\theta_i\}$ and $x'(\theta_i-) = f(x((\theta_i))), i \in \mathcal{A}$, where $x'(\theta_i-)$ is the left-sided derivative;*

(ii) *$x(\theta_i+) = J(x((\theta_i))$ for all θ_i.*

Theorem 2.1. *Assume that conditions $C1)-C6)$ hold. Then for every $x_0 \in D$ there exists an interval $(a, b) \subset \mathbb{R}, a < 0 < b$, such that the solution $x(t) = x(t, 0, x_0)$ of (1) exists on the interval.*

Definition 2.2. *A solution $x(t) : [a, \infty) \to \mathbb{R}^n, a \in \mathbb{R}$, of (1) is said to be continuable to ∞.*

Definition 2.3. *A solution $x(t) : (-\infty, b] \to \mathbb{R}^n, b \in \mathbb{R}$, of (1) is said to be continuable to $-\infty$.*

Definition 2.4. *A solution $x(t)$ of (1) is said to be continuable on $\mathbb{R}$ if it is continuable to ∞ and to $-\infty$.*

Definition 2.5. *A solution $x(t) = x(t, 0, x_0)$ of (1) is said to be continuable to a set $S \subset \mathbb{R}^n$ as time decreases (increases) if there exists a moment $\xi \in \mathbb{R}$, such that $\xi \le 0\, (\xi \ge 0)$ and $x(\xi) \in S$.*

Denote by $B(x_0, \xi) = \{x \in \mathbb{R}^n | ||x - x_0|| < \xi\}$ a ball with centre $x_0 \in \mathbb{R}^n$ and radius $\xi \in \mathbb{R}$.

The following Theorem provides sufficient conditions for the continuation of solutions of (1).

Theorem 2.2. *Assume that*

(a) *every solution $y(t, 0, x_0), x_0 \in D$, of*

$$y' = f(y). \tag{2}$$

satisfies the following conditions:

(a1) *it is continuable either to* ∞ *or to* Γ *as time increases,*

(a2) *it is continuable either to* $-\infty$ *or to* $\tilde{\Gamma}$ *as time decreases;*

(b) *for every* $x \in \tilde{\Gamma}$ *there exists a number* $\epsilon_x > 0$ *such that* $\bar{B}(x, \epsilon_x) \cap \Gamma = \emptyset$;

b′) *for every* $x \in \Gamma$ *there exists a number* $\tilde{\epsilon}_x > 0$ *such that* $\bar{B}(x, \tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x) \in \tilde{\Gamma} \times \mathbb{R}} \frac{\epsilon_x}{\sup_{\bar{B}(x,\epsilon_x)} \|f(x)\|} > 0$;

c′) $\inf_{(x,\tilde{\epsilon}_x) \in \Gamma \times R} \frac{\tilde{\epsilon}_x}{\sup_{B(x,\tilde{\epsilon}_x)} \|f(x)\|} > 0$.

Then every solution $x(t) = x(t, 0, x_0)$, $x_0 \in D$, *of (1) is continuable on* $\mathbb{R}$.

Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (1). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\theta \in \mathbb{R}$ and introduce a function $\psi(t) = x(t + \theta)$.

Lemma 2.1. *The set* $\{\theta_i - \theta\}$ *is a set of all solutions of the equation*

$$\Phi(\psi(t)) = 0. \tag{3}$$

The following condition is one of the main assumptions for DFs.

$C7)$ $\Gamma, \tilde{\Gamma} \subset \partial G$;

$\exists \epsilon > 0$ such that $\forall x \in \Gamma_\epsilon \cap G$ function $\Phi(x)$ is either positive or negative;

$\exists \epsilon > 0$ such that $\forall x \in \tilde{\Gamma}_\epsilon \cap G$ function $\tilde{\Phi}(x)$ is either positive or negative.

Lemma 2.2. *Assume that* $C1) - C7)$ *hold. Then* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* $x_0 \in D, t \in \mathbb{R}$.

Lemma 2.3. *If* $x(t) : T \to \mathbb{R}^n$ *is a solution of (1) then* $x(t + \theta), \theta \in \mathbb{R}$, *is also a solution of (1).*

Lemmas 2.1-2.3 imply that the following theorem is valid.

Theorem 2.3. *Assume that conditions* $C1) - C7)$ *are fulfilled. Then*

$$x(t_2, x(t_1, x_0)) = x(t_2 + t_1, x_0), \tag{4}$$

for all $t_1, t_2 \in \mathbb{R}$.

Let $x^0(t) : [a, b] \to \mathbb{R}^n, a \leq 0 \leq b$, be a solution of (1), $x^0(t) = x(t, 0, x_0), \theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are the points of discontinuity of $x^0(t)$, such that $a \le \theta_{-k} < \cdots < \theta_{-1} \le 0 \le \theta_1 < \cdots < \theta_m \le b$. Denote by $x(t) = x(t,0,\bar{x})$ another solution of (1).

Definition 2.6. *The solution $x(t) : [a,b] \to \mathbb{R}^n$ is said to be in an ϵ-neighbourhood of $x^0(t)$ if:*

1. *every point of discontinuity of $x(t)$ lies in an ϵ-neighbourhood of a point of discontinuity of $x^0(t)$;*
2. *For each $t \in [a,b]$ which is outside of ϵ-neighbourhood of points of discontinuity of $x^0(t)$ the inequality $\left\|x^0(t) - x(t)\right\| < \epsilon$ holds.*

Definition 2.7. *Hausdorff's topology, which is built on the basis of all ϵ-neighbourhoods, $0 < \epsilon < \infty$, of piecewise solutions will be called $B_{[a,b]}$-topology.*

Theorem 2.4. *Assume that conditions $C1) - C7)$ are satisfied. Then the solution $x(t)$ continuously depends on initial value in $B_{[a,b]}$ topology .*
Moreover, if all $\theta_i, i = -k,\ldots,-1,1,\ldots,m$, are interior points of $[a,b]$, then, for sufficiently small $||x_0 - \bar{x}||$, the solution $x(t) = x(t,0,\bar{x}), x(t) : [a,b] \to \mathbb{R}^n$, meets the surface Γ exactly $m+k-1$ times.

Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior. Denote by $x_j(t), j = \overline{1,n}$, a solution of (1) such that $x_j(t_0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0), \xi \in \mathbb{R}$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let θ_i^j be the moments of discontinuity of $x_j(t)$. By Theorem 2.4, for sufficiently small $|\xi|$ the solution $x_j(t)$ is defined on $[a,b]$.

Definition 2.8. *The solution $x^0(t)$ is said to be differentiable in $x_j^0, j = \overline{1,n}$, if*
A) there exist such constants $\nu_{ij}, i = -k,\ldots,-1,1,\ldots,m$, that

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|); \tag{5}$$

B) for all $t \in [a,b] \setminus \cup_{i=-k}^{m} (\theta_i, \hat{\theta}_i^j]$, the following equality is satisfied

$$x_j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \tag{6}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i, i = -k,\ldots,-1,1,\ldots,m$.

The pair $\{u_j, \{\nu_{ij}\}_i\}$ *is said to be a* $B-$ *derivative of* $x^0(t)$ *in initial value* x_0^j *on* $[a, b]$.

The following theorem is valid.

Theorem 2.5. *Assume that conditions* $C1) - C7)$ *are satisfied. Then the solution* $x^0(t)$ *of (1) has* $B-$ *derivatives in the initial value on* $[a, b]$.

3. **The B-smooth discontinuous flow**

Let $G \subset \mathbb{R}^n$ be an open set and $\Gamma, \tilde{\Gamma}$ be disjoint subsets of $\bar{G}$. Denote $D = G \cup \Gamma \cup \tilde{\Gamma}$.

Definition 3.1. *We say that a* $B-$ *smooth* DF *is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the following properties:*

$I)$ *The group property:*

(i) $\phi(0, x) : D \to D$ *is the identity;*

(ii) $\phi(t, \phi(s, x)) = \phi(t + s, x)$, *is valid for all* $t, s \in \mathbb{R}$ *and* $x \in D$.

$II)$ *If* $x \in D$ *is fixed then* $\phi(t, x) \in \mathcal{PC}^1(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t, x)$.

$III)$ *The function* $\phi(t, x)$ *is* $B-$ *differentiable in* $x \in D$ *on* $[a, b] \subset \mathbb{R}$ *for every* $\{a, b\} \subset \mathbb{R}$, *assuming that all discontinuity points of* $\phi(t, x)$ *are interior points of* $[a, b]$.

One can see that the system (1) defines a $B-$ smooth DF provided conditions $C1) - C7)$ and the conditions of the continuation theorem are fulfilled.

Definition 3.2. *We say that a* DF *is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the property* $I)$ *of Definition 3.1 and the following conditions are valid:*

$IV)$ *If* $x \in D$ *is fixed then* $\phi(t, x) \in \mathcal{PC}(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t, x)$.

$V)$ *The function* $\phi(t, x)$ *is* $B-$ *continuous in* $x \in D$ *on* $[a, b] \subset \mathbb{R}$ *for every* $\{a, b\} \subset \mathbb{R}$.

Comparing definitions of the $B-$ differentiability and the $B-$ continuity one can conclude that every $B-$ smooth DF is a DF.

Example 3.1. *Consider the following model for simple neural nets from* [3]. *We have modified its form according to the proposed equation (1).*

$$x_1' = x_2,\, x_2' = -\beta^2 x_1,\, p' = -\gamma p + x_1 + B_0, if\ (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$

$$x_1(t+) = x_2(t-),\, x_2(t+) = x_2(t-),\, p(t+) = 0, if\ x(t) \in \Gamma \wedge t \geq 0,$$

$$x_1(t-) = x_1(t+),\, x_2(t-) = x_2(t+),\, p(t-) = r, if\ x(t) \in \tilde{\Gamma} \wedge t \leq 0,$$

where $\beta, B_0 \in \mathbb{R}$ *are constants,* $\Gamma = \{(x_1, x_2, p)|\, p = r\}, \tilde{\Gamma} = \{(x_1, x_2, p)|\, p = 0\}, \Phi(x) = p - r,\, f(x) = (x_2, \beta^2 x_1, -\gamma p + x_1 + B_0), J(x) = (x_1, x_2, r), \beta, \gamma, r > 0,$ *are constants. We assume that* $G = \{(x_1, x_2, p)|0 < p < r, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$. *In the system the variable* $p(t)$ *is a scalar input of a neural trigger and* x_1, x_2, *are other variables. The value of* r *is the threshold. One can verify that the functions and the sets satisfy* $C1) - C7)$ *and the conditions of Theorem 2.2. That is, the system defines a DF.*

Remark 3.1. *The extended version of the paper has been submitted to Mathematical and Computer Modelling.*

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