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FEEDBACK DIFFERENTIAL SYSTEMS: APPROXIMATE AND LIMITING TRAJECTORIES

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Abstract. A "feedback differential system" is defined as a (generally discontinuous) parameterized differential inclusion (in particular, differential equation), that usually appears in the description of the complete solution of an optimal control problem or a differential game. In this article one obtains certain invariant characterizations of the uniform limits of two types of approximate trajectories: the well-known "Euler polygonal lines" and the less known "Isaacs approximate trajectories" suggested by the natural assumption of the discrete (step-by-step) "action" of a player in optimal control and differential games. The main results state that under very general hypotheses on the data, the limiting Euler and, respectively, Isaacs-Krassovskii-Subbotin trajectories are Carathéodory solutions of two distinct associated differential inclusions defined by corresponding "u.s.c.-convexified" limits of the original orientor fields. In particular, one provides a counter-example of a "conjecture" in Krassovskii and Subbotin(1974) and one gives a complete proof of the correct variant of this conjecture.

1. Introduction

The aim of this paper is to obtain certain "invariant" characterizations of the uniform limits of the well-known "Euler polygonal lines" in the general theory of Ordinary Differential Equations (ODE), on one hand and, on the other hand, of the less known "Isaacs approximate trajectories" of "proper" feedback differential

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systems, naturally appearing in the description of the complete solutions of optimal control problems and differential games.

We recall that the "limiting Euler trajectories" are frequently used not only in the general theory of ODE (e.g. Kurzweil(1986)[9]) and differential inclusions (e.g. Aubin and Cellina(1984)[1]) to prove the existence of classical (Newton or Carathéodory) solutions but also in the description of corresponding numerical algorithms; in a more general setting of certain types of differential inclusions, the study of the limiting Euler trajectories has been recently taken-up in Clarke et al(1998)[3].

On the other hand, the limiting "Isaacs-Krassovskii-Subbotin" trajectories have been considered first in Krassovskii and Subbotin(1974)[8] in an attempt to put on a more rigorous basis the rather heuristical approach in Isaacs(1965)[7] which referred only to the corresponding "approximate" trajectories; however, Krassovskii and Subbotin(1974), without mentioning Isaacs' name, identified "Isaacs approximate trajectories" as "Euler polygonal lines" (which is not true, as the definitions below show) and, moreover, formulated a "conjecture" (contradicted by the counterexample in Remark 4.3 below) according to which these limiting trajectories are Carathéodory solutions of a certain associated differential inclusion.

The main results of this paper are contained in Sections 3 and 4 below and show that under some mild hypotheses on the data, these types of limiting trajectories are *Carathéodory solutions* of certain associated *u.s.c.-convexified* differential inclusions which are closely related to concepts introduced by Cesari(1983)[2], Filippov(1988)[5] and Mirică(1992)[10] in different contexts.

As a general idea, we point out that, as in [1], [2], [3], [11], etc., in the proofs of the main results in Sections 3 and 4 below, we shall use, in a more explicit manner, a string of arguments based, first, on the "compactness" [Theorem 0.3.4 in Aubin and Cellina(1984)[1]] (which may be considered a refinement of the well-known Arzelà-Ascoli theorem), then the so called Banach-Saks-Mazur Lemma in Functional Analysis (e.g. Yosida(1974)[15]) and, finally, some arguments from basic Measure Theory (e.g. Dunford and Schwartz(1958)[4]).

One may note also that the results in Sections 3 and 4 below may explain some of the apparent "anomalies" pointed out in [Clarke et al(1998)[3], Section 4.1.2] and the role of the usual hypotheses in the general theory of ODE.

The paper is organized as follows: in Section 2 we present the necessary notations, definitions and preliminary results needed in the sequel and in Section 3 we prove the main result and some comments concerning the characterization of the limiting Euler(**E-**) trajectories; in Section 4 we prove in the same way the more complicated analogous results regarding the limiting Isaacs-Krassovskii-Subbotin(**IKS**) trajectories.

2. **Notations, definitions and preliminary results**

In this paper we shall be concerned mainly with a **feedback differential system** (actually a "parameterized differential inclusion") of the form

$$x' \in F(t,x) := f(t,x,U(t,x)),\ x(t_0) = x_0,\ t \in I = [t_0,t_1], \tag{2.1}$$

defined by a non-empty set (of "control parameters") U, a *parameterized vector field* $f(.,.,.) : D \times U \to R^n$ and a *multifunction* ("feedback strategy") $U(.,.) : D \subseteq R \times R^n \to \mathcal{P}(U)$ where $\mathcal{P}(U)$ denotes the family of all subsets of U; we note that in the particular case in which $U \subseteq R^n,\ f(t,x,u) \equiv u$ one has a "general" (non-parameterized) differential inclusion

$$x' \in F(t,x) := U(t,x),\ x(t_0) = x_0,\ t \in I = [t_0,t_1],\ (t_0,x_0) \in D \tag{2.2}$$

while in the case the multifunction $U(.,.)$ is either absent or "reduces" to a point, $U(t,x) \equiv \{u_0\}$, for some fixed point $u_0 \in U$, the inclusion in (2.1) becomes an "ordinary differential equation"

$$x' = g(t,x) := f(t,x,u_0),\ x(t_0) = x_0,\ (t_0,x_0) \in D. \tag{2.3}$$

In what follows, a $\Delta -$ *approximate solution* is related to a *partition ("division") of the interval* $I = [t_0,t_1]$ denoted by $\Delta = \{\tau^j; j \in \{0,1,...k+1\}\}$ where $t_0 = \tau^0 < \tau^1 < ... \tau^k < \tau^{k+1} = t_1$ and whose "norm"(or "mesh size") is defined by $|\Delta| := max\{\tau^{j+1} - \tau^j;\ 0 \le j \le k\}$.

Definition 2.1. If $\Delta = \{\tau^j; j = \{0, 1, ...k + 1\}\}$ is a partition of the interval $I = [t_0, t_1]$ then $x_\Delta(.) \in AC(I; R^n)$ is said to be:

(i) *an Euler* $\Delta-$ *approximate solution* of (2.1) if there exists a finite subset, $\{v^j;\ j \in \{0, 1, ...k\}\} \subset R^n$ such that:

$$x_\Delta(t_0) = x_\Delta(\tau^0) = x_0,\ v^j \in F(\tau^j, x_\Delta(\tau^j)),\ j \in \{0, 1, ...k\} \tag{2.7}$$

$$x_\Delta(t) = x_\Delta(\tau^j) + (t - \tau^j)v^j\ \forall\ t \in I^j = [\tau^j, \tau^{j+1}]; \tag{2.8}$$

(ii) *an Isaacs* $\Delta-$*approximate solution* if there exists a finite subset, $\{u^j;\ j \in \{0, 1, ...k\}\} \subset U$ such that the mappings $f(., x_\Delta(.), u^j)$ are (Lebesgue) integrable and satisfy the following relations:

$$x_\Delta(t_0) = x_\Delta(\tau^0) = x_0,\ u^j \in U(\tau^j, x_\Delta(\tau^j)),\ j \in \{0, 1, ...k\} \tag{2.9}$$

$$x_\Delta(t) = x_\Delta(\tau^j) + \int_{\tau^j}^{t} f(s, x_\Delta(s), u^j)ds\ \forall\ t \in I^j = [\tau^j, \tau^{j+1}]. \tag{2.10}$$

Remark 2.2. We note first that the mappings $x_\Delta(.)$ in (2.8), (2.10) are defined "recurrently" on the sub-intervals $I^j = [\tau^j, \tau^{j+1}] \subset I,\ j \in \{0, 1, ...k\}$ starting from the initial value $x_\Delta(\tau^0) = x_0$ and choosing, at each step, a point $v^j \in F(\tau^j, x_\Delta(\tau^j))$ (respectively, $u^j \in U(\tau^j, x_\Delta(\tau^j))$); moreover, on each sub-interval I^j the mapping $x_\Delta(.)$ in (2.10) is a Carathéodory solution of the O.D.E.

$$x'(t) = f(t, x(t), u^j)\ a.e.(I^j),\ j \in \{0, 1, ...k\},\ I^j = [\tau^j, \tau^{j+1}] \tag{2.11}$$

while the corresponding mapping in (2.8) is a *piecewise affine mapping* with the constant derivative $v^j \in F(\tau^j, x_\Delta(\tau^j))$ on the sub-interval $Int(I^j) = (\tau^j, \tau^{j+1})$; one may note that while an Euler $\Delta-$*solution* may be defined for general (non-parameterized) differential inclusions, the Isaacs $\Delta-$ *solutions* in (2.10) are specific to the "properly parameterized" differential inclusions in (2.1) since in the case of the general ones in (2.2), they become Euler $\Delta-$ *solutions*.

Further on, as it is well known, the "integrability" condition in Def.1(ii) (which is rather difficult to verify in the general case) is implied by the fact that the mappings $f(., ., u),\ u \in U$ are *Carathéodory vector fields* (e.g. [5], [9], [11], etc.); for the proof of the main result in Section 3 below we need the following more restrictive property:

Hypothesis 2.3. *The data of the problem (2.1) have the following properties:*

(i): $U \neq \emptyset,\ D = Int(D) \subseteq R \times R^n$ *(i.e. is open) and* $U(.,.) : D \to \mathcal{P}(U)$ *has non-empty values at each point;*

(ii): the mapping $f(.,.,.) : D \times U \to R^n$ *is such that there exists a null subset* $I_f \subset pr_1 D$ *such that:*

(ii_1)*: the mappings* $f(.,x,u),\ x \in pr_2 D,\ u \in U$ *are measurable;*

(ii_2)*: the mappings* $f(t,.,u),\ t \in pr_1 D \setminus I_f,\ u \in U$ *are continuous;*

(iii): the multifunctions $F(.,.) := f(.,.,U(.,.))$ *and* $U(.,.)$ *are "jointly" locally integrably-bounded in the sense that for any compact subset* $D_0 \subset D$ *there exists an integrable mapping* $c(.) \in L^1(pr_1 D_0; R_+)$ *and a null subset* $I_0 \subset pr_1 D_0$ *such that:*

$$||f(t,x,u)|| \leq c(t)\ \forall\ (t,x) \in D_0,\ t \in pr_1 D_0 \setminus I_0,\ u \in U(D_0) \tag{2.12}$$

where $U(D_0) := \bigcup\{U(s,y);\ (s,y) \in D_0\}$.

One may note that property (iii) is implied by the usual hypothesis according to which U is a compact topological space and $f(.,.,)$ is continuous (with respect to all variables); moreover, property (ii) implies the fact that $f(.,.,u),\ u \in U$ are Carathéodory vector fields hence the definition of the Isaacs $\Delta -$ *solutions* in Def.2.4(ii) makes sense without the "artificial" requirement of the integrability condition.

On the other hand, for the study of the limiting Euler-trajectories in Section 3 we need only a simpler "local boundedness" property of the orientor field $F(.,.)$ (see Th.3.1 below).

In what follows we shall study the corresponding types of "limiting trajectories" defined as "uniform limits" of the approximate trajectories in Def.2.1; we recall that the "limiting Euler trajectories" are frequently used not only in the general theory of ODE (e.g. Kurzweil(1986)[9]) and differential inclusions (e.g. Aubin and Cellina(1984)[1]) to prove the existence of classical (Newton or Carathéodory) solutions but also in the description of certain numerical algorithms; on the other hand, the limiting "Isaacs-Krassovskii-Subbotin" (**IKS-** trajectories have been considered first in Krassovskii and Subbotin(1974)[8] in an attempt to put on a more rigorous basis the rather heuristical approach in Isaacs(1965)[7] which referred only to the corresponding "approximate" trajectories.

Definition 2.4. The continuous mapping $x(.) \in C(I; R^n)$ is said to be:

(i): *an Euler*(**E**)*-trajectory* of the problem in (2.1) if there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0, 1, ...k_m + 1\}\},\ m \in N$ of the interval $I = [t_0, t_1]$, the subsets $\{v_m^j;\ j \in \{0, 1, ...k_m\}\} \subset R^n$ and the corresponding Euler $\Delta_m - solutions,\ x_m(.) := x_{\Delta_m}(.),\ m \in N$ in (2.7),(2.8) such that:

$$|\Delta_m| \to 0,\ x_m(t) \to x(t)\ uniformly\ for\ t \in I\ as\ m \to \infty; \qquad (2.13)$$

(ii): *an Isaacs-Krassovskii-Subbotin*(**IKS**)*-trajectory* of the problem (2.1) if there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0, 1, ...k_m + 1\}\},\ m \in N$ of the interval $I = [t_0, t_1]$, the subsets $\{u_m^j;\ j \in \{0, 1, ...k_m\}\} \subset U$ and the corresponding $Isaacs\ \Delta_m - solutions,\ x_m(.) := x_{\Delta_m}(.),\ m \in N$ in (2.9), (2.10) such that the properties in (2.13) are satisfied.

Note that, in general the uniform limit of $x_m(.)$ (in the topology generated by the norm $\|x(.)\|_C := \max\{\|x(t)\|;\ t \in I\}$ of the space $C(I; R^n)$ of continuous mappings) need not be absolutely continuous (**AC**), not even a.e. differentiable; a sufficient condition for this property is given in the following [*compactness theorem* 0.3.4 in Aubin and Cellina[1]] which seems to be more suitable than the classical Arzelà-Ascoli theorem, in the study of Carathéodory-type differential inclusions and differential equations.

Theorem 2.5 (**compactness**). *Let X be a Banach space, let $I \subset R$ be an interval and let $\{x_m(.)\} \subset AC(I; X)$ be a sequence of* **AC** *mappings with the following properties:*

(i): for each $t \in I$ the subset $\{x_m(t);\ m \in N\} \subset X$ is relatively compact;

(ii): there exists an integrable function $c(.) \in L^1(I; R_+)$ such that

$$\|x'_m(t)\| \leq c(t)\ a.e.(I)\ \forall\ m \in N.$$

Then there exists a subsequence $\{x_{m_j}(.)\}$ *and a mapping* $x(.) \in AC(I;X)$ *such that*

(1): $x_{m_j}(.) \to x(.)$ *uniformly on each compact subset of* I*;*

(2): $x'_{m_j}(.) \to x'(.)$ *weakly in the space* $L^1(I;X)$ *of integrable mappings.*

One may note that in the case $X = R^n$, if $x_m(.)$ are *equally bounded* and property (ii) is satisfied then $x_m(.)$ are also uniformly equi-continuous hence one may apply the Arzelà-Ascoli Theorem but the conclusion in Theorem 2.5 is stronger, stating not only the fact that the limit is **AC** but also the weak convergence (in L^1) of the derivatives.

As in the study of many other problems (e.g. [11]), at a certain stage of the proofs of the main results, we shall use the following important theorem in Functional Analysis which seems to belong, jointly, to Banach, Saks and Mazur though in some books and monographs only one, two or no names are mentioned.

Theorem 2.6 (**Banach-Saks-Mazur**). *Let* X *be a normed space, let* X^* *be its dual and let* $x_m,\ x \in X,\ m \in N$ *be such that* $x_m \to x$ *weakly i.e. such that* $x^*(x_m) \to x^*(x)\ \forall\ x^* \in X^*$.

Then for each $m \in N$ *there exist the integer* $i_m \geq m$ *and the real numbers,* $c_m^i \in R$ *such that*

$$c_m^i \geq 0,\ \sum_m^{i_m} c_m^i = 1 \text{ and } \|y_m - x\| \to 0 \text{ if } y_m := \sum_m^{i_m} c_m^i x_i.$$

For the proof and equivalent statements of this important result we refer to Yosida [15], to Dunford and Schwartz [4] and to the references therein.

Finally, we shall use also the following result in Measure Theory which is very often used as a piece of "Mathematical folklore".

Theorem 2.7. (**Measure Theory**). *Let* X *be a Banach space, let* $I = [a,b] \subset R$ *be a compact interval and let* $x_m(.),\ x(.) \in L^1(I;X)$ *be such that* $x_m(.) \to x(.)$ *strongly in* L^1.

Then there exist a subsequence $x_{m_j}(.)$ *such that* $x_{m_j}(t) \to x(t)$ *a.e.*(I).

For a proof of this theorem we refer to Theorem 3.3.6 and Corollary 3.6.13 in Dunford and Schwartz [4].

In what follows $\|.\|$ denotes the *Euclidean norm* on R^n, if $r > 0$ and $x \in R^n$ then $B_r(x) := \{y \in R^n;\ \|y-x\| < r\}$ and if $A \subset R^n$ then $Int(A),\ Cl(A),\ co[A],\ \overline{co}[A]$ denote the interior, the closure, the convex hull and, respectively, the closed convex hull of A.

3. **Limiting Euler trajectories**

In this section we use Theorems 2.5, 2.6, 2.7 to obtain certain "invariant" characterizations of the *limiting Euler trajectories* in Def.2.4(i) and, in particular, of existence theorems for solutions of upper semicontinuous convex-valued differential inclusions and differential equations.

Theorem 3.1. *If the "orientor field" $F(.,.) : D = Int(D) \subseteq R \times R^n \to \mathcal{P}(R^n)$* **is locally bounded** *in the sense that for any compact subset $D_0 \subset D$ there exists $c > 0$ such that*

$$\|v\| \leq c \ \forall\ v \in F(D_0) := \bigcup\{F(t,x);\ (t,x) \in D_0\} \tag{3.1}$$

*and $x(.) \in C(I; R^n)$ is an Euler trajectory in the sense of Def.2.4(i) of the differential inclusion in (2.2) then $x(.)$ is a Carathéodory (**AC**) solution of the u.s.c.-convexified differential inclusion*

$$x' \in F^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[F((t-\delta, t+\delta) \times B_\delta(x))]. \tag{3.2}$$

Proof. From Def.2.4(i) it follows that there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0, 1, ...k_m + 1\}\},\ m \in N$ of the interval $I = [t_0, t_1]$, the subsets $\{v_m^j;\ j \in \{0, 1, ...k_m\}\} \subset R^n$ and the corresponding Euler $\Delta_m - solutions,\ x_m(.) := x_{\Delta_m}(.), m \in N$ in (2.7),(2.8), hence such that the following relations are satisfied on the intervals $I_m^j = [\tau_m^j, \tau_m^{j+1}]$:

$$x_m(t) = x_m(\tau_m^j) + (t - \tau_m^j)v_m^j, t \in I_m^j, v_m^j \in F(\tau_m^j, x_m(\tau_m^j)) \tag{3.3}$$

and such that the properties in (2.13) hold true; obviously, the property in (3.3) is equivalent with the fact that

$$x_m'(t) = v_m^j\ \forall\ t \in (\tau_m^j, \tau_m^{j+1}),\ j \in \{0, 1, ...k_m\},\ x(t_0) = x_0. \tag{3.4}$$

On the other hand, since $D \subseteq R \times R^n$ is open and $x(.)$ is continuous, there exists $r > 0$ and a rank $m_r \in N$ such that

$$D_0 := \{(t,y);\ t \in I,\ y \in \overline{B}_r(x(t)) := Cl(B_r(x(t))\} \subset D \tag{3.5}$$

$$(t, x_m(t)) \in D_0\ \forall\ t \in I,\ m \geq m_r. \tag{3.6}$$

As already stated, the general idea is to show that Theorems 2.5, 2.6, 2.7 are successively applicable to the above sequence $\{x_m(.);\ m \geq m_r\} \subset AC(I;R^n)$; to this end we note that from the fact that $(\tau_m^j, x_m(\tau_m^j)) \in D_0\ \forall\ m \geq m_r,\ j \in \{0,1,...k_m\}$ and from the properties in (3.1) and (3.4) it follows that

$$\|x'_m(t)\| \leq c\ \forall\ t \in I \setminus \{\tau_m^j;\ j \in \{0,1,...k_m\},\ m \geq m_r. \tag{3.7}$$

Therefore Th.2.5 is applicable to the sequence $\{x_m(.);\ m \geq m_r\}$ hence taking possibly a subsequence, without loss of generality, we may assume that $x(.) \in AC(I;R^n)$ and that $x'_m(.) \to x'(.)$ *weakly in* $L^1(I;R^n)$; next, we apply first Th.2.6 to obtain the existence of the non-negative numbers $c_m^i \geq 0$ and of the natural numbers $i_m \geq m$ such that

$$\sum_m^{i_m} c_m^i = 1,\ \|\sum_m^{i_m} c_m^i x'_i(.) - x'(.)\|_{L^1} \to 0\ as\ m \to \infty \tag{3.8}$$

while from Th.2.7 it follows that, taking possibly a subsequence, one may assume that there exists a null subset $I_2 \subset I$ such that:

$$c_m^i \geq 0,\ \sum_m^{i_m} c_m^i = 1,\ y_m(t) := \sum_m^{i_m} c_m^i x'_i(t) \to x'(t)\ \forall\ t \in I \setminus I_2. \tag{3.9}$$

From (2.13) it follows now that for each $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $\forall\ t \in I,\ m \geq m_\delta\ \exists j = j(t,m) \in \{0,1,...k_m\}$ such that $(\tau_m^j, x_m(\tau_m^j)) \in (t-\delta, t+\delta) \times B_\delta(x(t))$ which, in view of (3.4) and of the fact that $v_m^j \in F(\tau_m^j, x_m(\tau_m^j))$ implies:

$$x'_m(t) \in F((t-\delta, t+\delta) \times B_\delta(x(t))\ \forall\ t \in I \setminus \{\tau_m^j; j \in \{0,1,...k_m+1\}\}, m \geq m_\delta$$

and which, in turn, in view of (3.9), implies the fact that $x(.)$ is a Carathéodory solution of the differential inclusion in (3.2).

In the particular case of locally bounded but otherwise arbitrary vector fields in (2.3) we obtain immediately the following result.

Corollary 3.2. *If* $g(.,.) : D = Int(D) \subseteq R\times R^n \to R^n$ *is a vector field that is* **locally bounded** *in the sense of (3.1) and* $x(.) \in C(I; R^n)$ *is an Euler-trajectory of the ODE in (2.3) in the sense of Def.2.4(i) then* $x(.)$ *is a Lipschitzian (Carathéodory) solution of the differential inclusion*

$$x' \in g^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[g((t-\delta,t+\delta)\times B_\delta(x))]. \tag{3.10}$$

In particular, if $g(.,.)$ *is continuous (with respect to both variables) then* $x(.)$ *is a continuously differentiable ("Newton's") solution of the same equation.*

Remark 3.3. One may note that the statement in Cor.3.2 is much weaker than the corresponding one in Cor.4.2 below for **IKS**-trajectories of ODE and simple examples (e.g. [Clarke et al.[3], Section 4.1.2]) show that it cannot be significantly improved; according to these examples, even if $g(.,.)$ is continuous, an **E**-trajectory may not be a **C**-solution of (2.3) and, on the other hand, a Newton (i.e., of class C^1) solution may not be an **E**-trajectory. The only case in which an equivalence analogous to the one in Cor.4.2 may hold seems to be that of the "Peano-Lipschitz vector fields", $g(.,.)$, which are continuous with respect to both variables and locally-Lipschitz with respect to the second one or, slightly more general, that have the uniqueness property in the theory of ODE.

4. **Limiting Isaacs-Krassovskii-Subbotin trajectories**

The main result of this section is the following theorem giving the correct variant of the "conjecture" in [Krassovskii and Subbotin (1974)[8], Section 2.7].

Theorem 4.1. *If Hypothesis 2.3 is satisfied and* $x(.) \in C(I; R^n)$ *is a* **IKS**-*trajectory in the sense of Def.2.4(ii) then* $x(.)$ *is a Carathéodory solution of the u.s.c.-convexified differential inclusion*

$$\begin{aligned} x' \in F_u^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[\bigcup\{f(t,y,U(s,z));\ y\in B_\delta(x),\\ (s,z)\in(t-\delta,t+\delta)\times B_\delta(y)\}]. \end{aligned} \tag{4.1}$$

Proof. From Def.2.4(ii) it follows that there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j\in\{0,1,...k_m+1\}\},\ m\in N$, of the interval $I=[t_0,t_1]$, the subsets $\{u_m^j;\ j\in$

$\{0,1,...k_m\}\} \subset U$ and the corresponding Isaacs $\Delta_m-solutions,\ x_m(.) := x_{\Delta_m}(.), m \in N$ in (2.9),(2.10), hence such that:

$$x_m(t_0) = x_m(\tau^0) = x_0,\ u_m^j \in U(\tau_m^j, x_m(\tau_m^j)),\ j \in \{0,1,...k_m\} \tag{4.2}$$

$$x_m(t) = x_m(\tau_m^j) + \int_{\tau_m^j}^{t} f(s, x_m(s), u_m^j) ds\ \forall\ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}] \tag{4.3}$$

and such that the properties in (2.13) hold true; as already noted, the property in (4.3) is equivalent with the fact that there exists a null subset, $I_1 \subset I,\ I_1 \supset \{\tau_m^j;\ m \in N,\ j \in \{0,1,...k_m+1\}$ such that:

$$x_m'(t) = f(t, x_m(t), u_m^j)\ \forall\ t \in (\tau_m^j, \tau_m^{j+1}) \setminus I_1,\ x(t_0) = x_0. \tag{4.4}$$

On the other hand, since $D \subseteq R \times R^n$ is open and $x(.)$ is continuous (hence $x(I) \subset R^n$ is compact), there exist $r, m_r > 0$ such that (3.5) and (3.6) hold.

As in the proof of Th.3.1, the general idea is to show that Theorems 2.5, 2.6, 2.7 are successively applicable to the above sequence $\{x_m(.);\ m \geq m_r\} \subset AC(I; R^n)$; to this end we note that from Hypothesis 2.3(iii) it follows that for the compact subset $D_0 \subset D$ in (3.5) there exists an integrable function $c(.) \in L^1(I; R_+)$ and a null subset $I_0 \subset I = pr_1 D_0$ such that (2.12) holds; further on, since from (3.6) it follows that, in particular, $(\tau_m^j, x_m(\tau_m^j)) \in D_0$, hence $u_m^j \in U(\tau_m^j, x_m(\tau_m^j)) \subset U(D_0)$, from (2.12) and (4.4) it follows that:

$$\|x_m'(t)\| \leq c(t)\ \forall\ t \in I \setminus (I_0 \cup I_1),\ m \geq m_r. \tag{4.5}$$

Therefore Th.2.5 is applicable to the sequence $\{x_m(.);\ m \geq m_r\}$ hence taking possibly a subsequence (without loss of generality), we may assume that $x(.) \in AC(I; R^n)$ and also that $x_m'(.) \to x'(.)$ *weakly in* $L^1(I; R^n)$; next, we apply Th.2.6 to obtain the existence of the non-negative numbers $c_m^i \geq 0$ and of the natural numbers $i_m \geq m$ such that (3.8) holds while from Th.2.7 it follows that, taking possibly a subsequence, one may assume that there exists a null subset $I_2 \subset I,\ I_2 \supset I_1$, such that (3.9) holds.

We shall prove now that for each $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $\forall\ t \in I \setminus I_1,\ m \geq m_\delta$ one has:

$$x_m'(t) \in \bigcup \{f(t, x_m(t), U(s,z));\ (s,z) \in (t-\delta, t+\delta) \times B_\delta(x_m(t))\} \tag{4.6}$$

which, in view of (2.13) and (3.9), implies in the following way the fact that $x(.)$ is a Carathéodory solution of the differential inclusion in (4.1): from (2.13) it follows that for $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $x_m(t) \in B_\delta(x(t)) \ \forall \ t \in I, \ m \geq m_\delta$ hence from (4.6) it follows that for each $\delta > 0, \ t \in I \setminus I_1$ one has:

$$x'_m(t) \in \bigcup \{f(t, y, U(s, z)); y \in B_\delta(x(t)), (s, z) \in (t - \delta, t + \delta) \times B_\delta(y)\}$$

which, in view of (3.9), implies the fact that $x(.)$ is a **C**-solution of (4.1).

To prove (4.6) we use first the well-known *absolute continuity of the Lebesgue integral*, $J \mapsto \int_J c(s)ds$ to obtain that for $\delta > 0$ there exists $\eta_\delta > 0$ such that

$$\int_J c(s)ds < \delta \ \forall \ J \subset I, \ \mu(J) < \eta_\delta;$$

next, we use the property in (2.13) to obtain the existence of a rank $m_\delta \geq m_r$ such that

$$|\Delta_m| < \min\{\delta, \eta_\delta\}, \ \|x_m(t) - x(t)\| < \delta \ \forall \ m \geq m_\delta, \ t \in I$$

hence, in particular, such that:

$$\int_{\tau_m^j}^t c(s)ds < \delta, \ t - \tau_m^j \leq |\Delta_m| < \delta \ \forall \ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}].$$

Therefore from (3.3), (3.4) and (4.5) it follows that:

$$\|x_m(t) - x_m(\tau_m^j)\| \leq \int_{\tau_m^j}^t c(s)ds < \delta, \ \forall \ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}]$$

hence $(\tau_m^j, x_m(\tau_m^j)) \in (t - \delta, t + \delta) \times B_\delta(x_m(t))$ and the relation in (4.6) follows from (3.4) and from the fact that $u_m^j \in U(\tau_m^j, x_m(\tau_m^j))$.

In the particular case of the Carathéodory ODE in (2.3) we obtain the following result.

Corollary 4.2. *If $g(.,.) : D = Int(D) \subseteq R \times R^n \to R^n$ is a Carathéodory vector field in the sense of Hypothesis 2.3(ii),(iii) then $x(.) \in AC(I; R^n)$ is an* **IKS***-trajectory of the ODE in (2.3)* **iff** *it is a Carathéodory solution of the same equation.*

Proof. If $x(.) \in AC(I; R^n)$ is a Carathéodory(**C**) solution of (2.3) then, obviously, for any partition Δ of the interval I it is an Isaacs $\Delta -$ *solution* hence $x(.)$ is an **IKS**-trajectory in the sense of Def.2.4(ii).

Conversely, if $x(.) \in C(I;R^n)$ is a **IKS**-trajectory then according to Th.3.1 it is a **C**-solution of the differential inclusion in (4.1) which, in this case, is defined by the orientor field

$$F_u^{co}(t,x) = (g(t,.))^{co}(x) := \bigcap_{\delta>0} \overline{co}[g(t,B_\delta(x))];$$

finally, since $g(t,.)$ is assumed to be continuous for $t \in pr_1D \setminus I_g$ for some null subset $I_g \subset pr_1D$, it is easy to see that

$$F_u^{co}(t,x) = (g(t,.))^{co}(x) = \{g(t,x)\}\ \forall\ t \in pr_1D \setminus I_g, x \in pr_2D$$

hence $x(.)$ is a **C**-solution of (2.3).

Remark 4.3. We recall that the "conjecture" in [Krassovskii and Subbotin (1974)[8], Section 2.7] (in the case of the single-valued feedback strategies $U(.,.) = \{u(.,.)\}$), states that "using standard tools in the theory of ODE one may prove that any "perfect" (i.e. **IKS**) trajectory is a "generalized" trajectory in the sense that it is a Carathéodory solution of the associated differential inclusion" in (3.2).

Besides the fact that Theorems 2.5, 2.6, 2.7 above (that have been used essentially in the proof of Th.3.1, 4.1) may hardly be taken as "standard tools in the theory of ODE", the conjecture may be considered justified only in the case the multifunctions in (4.1) and (3.2) are related as follows: $F_u^{co}(t,x) \subseteq F^{co}(t,x)\ \forall\ (t,x) \in D$ as it is the case of the *vector fields* $g(.,.)$ in Cor.4.2, since one may write successively: $F_u^{co}(t,x) \equiv g(t,.)^{co}(x) \equiv \{g(t,x)\} \subseteq F^{co}(t,x) \equiv g^{co}(t,x)$; in the general case, the orientor fields in (4.1) and (3.2) may not be related in this way and very simple examples show that Krassovskii-Subbotin conjecture is false. For instance, if $d(.)$ is the well-known "Dirichlet function"

$$d(t) := \begin{cases} 1\ if\ t \in Q \\ 0\ if\ t \in R \setminus Q \end{cases}$$

and $f(t,x,u) \equiv d(t) + u,\ U(t,x) \equiv \{1 - d(t)\} \subset U := [0,1]$ then obviously $F(t,x) \equiv F^{co}(t,x) \equiv \{1\}$ while the convexified u.s.c.-limit in (4.1) is given by: $F_u^{co}(t,x) \equiv d(t) + [0,1]$; therefore, the only **C**-solution of (3.2) that satisfies $x(0) = 0 \in R$ is the function $x(t) = t,\ t \in I = [0,1]$ while taking a sequence $\{\Delta_m\}$ of partitions of

$I = [0, 1]$ such that $\tau_m^j \in Q$ it follows that $x_0(t) \equiv 0$ is an **IKS**-trajectory of (2.1) that it is not a **C**-solution of (3.2).

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