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DIFFERENTIAL SUBORDINATION AND STARLIKENESS OF ANALYTIC FUNCTIONS

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Abstract. In the present paper by using the method of differential subordination we aim to prove some classical results in univalent function theory. In particular we give some new sufficient condition for an analytic function to be starlike and convex in the unit disc U. Also by applying Ruscheweyh derivative we investigate some argument properties of some subclasses of univalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For f and g which are analytic in U, we say that f is subordinate to g,written $f(z) \prec g(z)$, if there exists an analytic function ω in U such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$.

For $0 < b \leq a$, the function $p \in \mathcal{A}$ is said to be in $P(a, b)$ if and only if

$$|p(z) - a| < b, \qquad z \in U.$$

Without loss of generality we omit the trivial case $p(z) = 1$ and assume that $|1-a| < b$.

For $-1 \leq B < A \leq 1$, the function $p \in \mathcal{A}$ is said to be in $P[A, B]$ if and only if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad z \in U.$$

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Here the symbol $' \prec '$ stands for subordination.For $0 < b \leq a$, there is a correspondence between $P(a,b)$ and $P[A,B]$, namely

$$P(a,b) \equiv P\left[\frac{b^2 - a^2 + a}{b}, \frac{1-a}{b}\right].$$

Two subclasses of $P(a,b)$ and $P[A,B]$ that have been studied extensively by other authors (e.g.see [2])are $P(1,b)$ and $P[A,-1]$.

The object of the present paper is to investigate some argument properties of analytic functions. We also obtain new sufficient condition for starlikeness and convexity.

First we introduce a subordination criterion for $p(z)$ which is subordinate to $\left(\frac{1+z}{1-z}\right)^{\eta}$.

To establishing our main results,we shall need the following results, which are due to Miller and Mocanu [4], Nunokawa [5] and Miller and Mocanu [4], respectively.

Lemma 1.1. *Let h be a convex function in U and let λ be analytic in U with $\Re\lambda(z) \geq 0$. If q is analytic in U and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z),$$

implies

$$q(z) \prec h(z) \quad (z \in U).$$

Lemma 1.2. *Let q be analytic in U with $q(0) = 0$ and $q(z) \neq 0$ in U. Suppose that there exists a point $z_0 \in U$ such that*

$$|\arg q(z)| < \frac{\pi\eta}{2} \text{ for } |z| < |z_0| \tag{1}$$

and

$$|\arg q(z_0)| = \frac{\pi\eta}{2}, \tag{2}$$

where $0 < \eta \leq 1$. Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\eta, \tag{3}$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{\pi\eta}{2}, \tag{4}$$

$$k \leq \frac{-1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{-\pi\eta}{2}, \tag{5}$$

and

$$q(z_0)^{\frac{1}{\eta}} = \pm ia \quad (a > 0). \tag{6}$$

Lemma 1.3. *Let F be analytic in U and let G be analytic and univalent on $\bar{U}$, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in U$ and $\xi_0 \in \partial U$, and $m \geq 1$ for which $F(|z| < |z_0|) \subset G(|z| < |z_0|), F(z_0) = G(\xi_0)$ and $z_0 F'(z_0) = m\xi_0 G'(\xi_0)$.*

2. **Main Results**

We now state and prove our main results.

Lemma 2.1. *Let p be analytic in U with $p(0) = 1$. If*

$$\left|\arg\left[p(z) + \frac{\lambda}{S(z)} zp'(z)\right]\right| < \frac{\pi}{2}\delta \quad (0 < \delta \leq 1, \lambda \geq 0),$$

for some $S(z)$ where $S(z) \in P(a,b)$, then

$$|\arg p(z)| < \frac{\pi}{2}\eta,$$

where $\eta \quad (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right). \tag{7}$$

Proof. Let $h(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, we observe that h is convex and $h(0) = 1$. Applying Lemma 1.1 for this h with $\lambda(z) = \frac{\lambda}{S(z)}$, we see that $\Re p(z) > 0$ in U and hence $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that the conditions (1) and (2) are satisfied, then (by Lemma 1.2) we obtain (3) under the restrictions (4),(5) and (6). Since $S(z) \in P(a,b)$ we have

$$S(z) = re^{i\frac{\pi}{2}\phi},$$

where $a - b < r < a + b$ *and* $\frac{-2}{\pi}\sin^{-1}(\frac{b}{a}) < \phi < \frac{2}{\pi}\sin^{-1}(\frac{b}{a})$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia \quad (a > 0)$, we obtain

$$\begin{aligned}
\arg\left[p(z_0) + \frac{\lambda}{S(z_0)} z_0 p'(z_0)\right] &= \arg p(z_0) + \arg\left(1 + \frac{\lambda}{S(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right) \\
&= \frac{\pi}{2}\eta + \arg\left(1 + \frac{\lambda}{re^{i\frac{\pi}{2}\phi}} i\eta k\right) \\
&= \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta k \cos(\frac{\pi}{2}\phi)}{r + \lambda\eta k \sin(\frac{\pi}{2}\phi)}\right) \\
&\geq \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta \cos(\sin^{-1}(\frac{b}{a}))}{a + b + \lambda\eta\frac{b}{a}}\right) \\
&= \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right) \\
&= \frac{\pi}{2}\delta
\end{aligned}$$

This is a contradiction to the assumption of our lemma.
Next, suppose that $p(z_0) = -ia \quad (a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
\arg\left[p(z_0) + \frac{\lambda}{S(z_0)} z_0 p'(z_0)\right] &= \frac{-\pi}{2}\eta - \tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right) \\
&= \frac{-\pi}{2}\delta,
\end{aligned}$$

where δ is given by (7) which contradict the assumption. This completes the proof of our lemma.

Theorem 2.2. *Let η be as defined by (7). Let $M(z) = z^n + ...$ and $N(z) = z^n + ...$ be analytic in U and such that, N satisfies*

$$\frac{zN'(z)}{N(z)} \in P(a, b).$$

Then

$$\left|\arg\left[(1 - \lambda)\frac{M(z)}{N(z)} + \lambda\frac{M'(z)}{N'(z)}\right]\right| < \frac{\pi}{2}\delta.$$

implies

$$\left|\arg\frac{M(z)}{N(z)}\right| < \frac{\pi}{2}\eta.$$

Proof. Consider the function $p(z) = \frac{M(z)}{N(z)}$ and let $S(z) = \frac{zN'(z)}{N(z)}$. Then by hypothesis, p is analytic and $p(0) = 1$. Hence all the conditions of Lemma 2.1 are satisfied. Now it is elementary to show that

$$(1-\lambda)\frac{M(z)}{N(z)} + \lambda\frac{M'(z)}{N'(z)} = p(z) + \frac{\lambda}{S(z)}zp'(z).$$

And hence Theorem 2.2 follows from Lemma 2.1.

The $\nu - th$ order Ruscheweyh Derivative [6] D^ν of a function $f \in \mathcal{A}$ is defined by

$$D^\nu f(z) = \frac{z}{(1-z)^{1+\nu}} * f(z) = z + \sum_{k=2}^{\infty} B_k(\nu)a_k z^k,$$

where

$$B_k(\nu) = \frac{(1+\nu)(2+\nu)...(\nu+k-1)}{(k-1)!}.$$

The operator $'*'$ stands for the convolution or Hadamard product of two power series $f(z) = \sum_{i=1}^{\infty} a_i z^i$ and $g(z) = \sum_{i=1}^{\infty} b_i z^i$ defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{i=1}^{\infty} a_i b_i z^i.$$

From the definition of D^ν and the properties of convolution $'*'$ follows the identity

$$z(D^\nu f(z))' = (1+\nu)D^{1+\nu}f(z) - \nu D^\nu f(z). \tag{8}$$

Corollary 2.3. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^\nu f(z)}{D^\mu g(z)} + \lambda\frac{(D^\nu f(z))'}{(D^\mu g(z))'}\right]\right| < \frac{\pi}{2}\delta, \quad (\nu \geq 0, \mu \geq 0, \lambda \geq 0, 0 < \delta \leq 1)$$

for some g where $D^\mu g(z) \in P[A,B], (-1 < B < A \leq 1)$, then

$$\left|\arg\frac{D^\nu f(z)}{D^\mu g(z)}\right| < \frac{\pi}{2}\eta,$$

where $\eta, (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda\eta\cos\left(\sin^{-1}\frac{A-B}{1-AB}\right)}{\frac{1+A}{1+B} + \lambda\eta\frac{A-B}{1-AB}}\right). \tag{9}$$

Proof. If we let $a = \frac{1-AB}{1-B^2}.b = \frac{A-B}{1-B^2}, M(z) = D^\nu f(z)$ and $N(z) = D^\mu g(z)$ then in this case η is given by (9) and the corollary now follows from Theorem 2.2.

Taking $B \mapsto A$ and $g(z) = z$ in Corollary 2.3, we have

Corollary 2.4. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu}f(z)}{z}+\lambda(D^{\nu}f(z))'\right]\right|<\frac{\pi}{2}\delta, \quad (\nu\geq 0,\lambda\geq 0, 0<\delta\leq 1),$$

then

$$\left|\arg\frac{D^{\nu}f(z)}{z}\right|<\frac{\pi}{2}\eta,$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}tan^{-1}\lambda\eta.$$

By using the same technique as in the proof of Lemma 2.1, we obtain

Theorem 2.5. *Let $f \in \mathcal{A}$.If*

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu+1}f(z)}{z}+\lambda(D^{\nu+1}f(z))'\right]\right|<\frac{\pi}{2}\delta, \quad (\nu\geq 0,\lambda\geq 0, 0<\delta\leq 1),$$

then

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu}f(z)}{z}+\lambda(D^{\nu}f(z))'\right]\right|<\frac{\pi}{2}\eta,$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}tan^{-1}\left(\frac{\eta}{1+\nu}\right).$$

Corollary 2.6. *Let $f \in \mathcal{A}$. If*

$$|\arg(f'(z)+\lambda zf''(z))|<\frac{\pi}{2}\delta,$$

then

$$\left|\arg\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right]\right|<\frac{\pi}{2}\eta$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}\tan^{-1}\eta.$$

We note that, by making use of Theorem 2.3, one can construct several new results for Bazilevic functions.

Theorem 2.7. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu}f(z)}{z}+\lambda\frac{(D^{\nu}f(z))'}{z}\right]\right| < \frac{\pi}{2}\delta, \quad (\nu > 0, \lambda \geq 0, 0 < \delta \geq 1)$$

then

$$\left|\arg\frac{D^{\nu}f(z)}{D^{\nu-1}f(z)}\right| < \frac{\pi}{2}(\eta_1+\eta_2),$$

where η_1, η_2 are the solutions of the equations

$$\delta = \eta_1 + \frac{2}{\pi}\tan^{-1}\lambda\eta_1, \tag{10}$$

and

$$\delta = \eta_2 + \frac{2}{\pi}tan^{-1}\lambda\eta_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta_2+\tan^{-1}\lambda\eta_2}{\nu}\right). \tag{11}$$

Proof. Using Corollary 2.4 and Theorem 2.5, we obtain

$$\left|\arg\frac{D^{\nu}f(z)}{z}\right| < \frac{\pi}{2}\eta_1, \tag{12}$$

and

$$\left|\arg\frac{D^{\nu-1}f(z)}{z}\right| < \frac{\pi}{2}\eta_2, \tag{13}$$

where η_1 and η_2 are defined by (10) and (11). Hence by using (12) and (13) we get our result.

Letting $\nu = 1$ and $\lambda = 1$ in Theorem 2.7 we have

Corollary 2.8. *Let $f \in \mathcal{A}$. If*

$$|\arg(f'(z)+zf''(z))| < \frac{\pi}{2}\delta, \qquad (0 < \delta \leq 1)$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}(\eta_1+\eta_2)$$

where η_1 and η_2 are the solutions of the equations

$$\delta = \eta_1 + \frac{2}{\pi}\tan^{-1}\eta_1,$$

and

$$\delta = \eta_2 + \frac{2}{\pi}\tan^{-1}\eta_2 + \frac{2}{\pi}\tan^{-1}(\eta_2+\tan^{-1}\eta_2).$$

Lemma 2.9. *Let λ be a function defined on U satisfies*

$$\eta = \inf_{z\in U}\left(\Re\lambda(z) - \cot\frac{\pi}{2}\delta|\Im\lambda(z)|\right) > 0, \tag{14}$$

and let

$$\beta(\eta,\delta) = (t_0)^{\delta}\left[\cos\frac{\pi}{2}\delta - \frac{\delta\eta}{2}\sin\frac{\pi}{2}\delta(t_0 + \frac{1}{t_0})\right], \tag{15}$$

be such that $2\beta(\eta,\delta) + \eta \geq 0$ with $t_0 = \frac{\cot\frac{\pi}{2}\delta + \sqrt{\cot^2\frac{\pi}{2}\delta + \eta^2(1-\delta^2)}}{\eta(1+\delta)}$.
If p be analytic in U with $p(0) = 1$,satisfies

$$\Re[p(z) + \lambda(z)zp'(z)] > \beta(\eta,\delta)$$

then

$$|\arg p(z)| < \frac{\pi}{2}\delta.$$

Proof. Let $h(z) = (\frac{1+z}{1-z})^{\delta}$, we observe that h is convex and $h(0) = 1$. Applying Lemma 1.1 for this h with $\lambda(z)$, we see that $\Re p(z) > 0$ and hence $p(z) \neq 0$ in U. For completing the proof of lemma we need only to show that $p(z) \prec h(z)$. If $p(z)$ is not subordinate to h, then by Lemma 1.3 there exist points $z_0 \in U$ and $\xi_0 \in \partial U$, and $m \geq 1$ such that

$$p(|z| \subset |z_0|) \subset q(U), p(z_0) = q(\xi_0) \quad \text{and} \quad z_0 p'(z_0) = m\xi_0 q'(\xi_0).$$

Since $p(z_0) \neq 0, \xi_0 \neq \pm 1$, by letting X and Y be the real and imaginary part of $\lambda(z_0)$,from (14), we find that

$$X + \cot\frac{\pi}{2}\delta Y \geq X - \cot\frac{\pi}{2}\delta|Y| \geq \eta > 0,$$

and

$$X - \cot\frac{\pi}{2}\delta Y \geq X - \cot\frac{\pi}{2}\delta|Y| \geq \eta > 0. \tag{16}$$

Further if we put $ix = \frac{1+\xi_0}{1-\xi_0}$ and use the above observations, we obtain

$$p(z_0) + \lambda(z_0)z_0 p'(z_0) = (ix)^{\delta}\left[1 + i\frac{m\delta}{2}(X + iY)\frac{1+x^2}{x}\right].$$

For $x \neq 0$,

$$\Re(p(z_0)+\lambda(z_0)z_0p'(z_0))$$

$$= \Re \quad \begin{cases} |x|^\delta \left(\cos\frac{\pi}{2}\delta + i\sin\frac{\pi}{2}\delta\right)\left(1-\frac{m\delta}{2}(Y-iX)\frac{1+x^2}{|x|}\right) & if x>0 \\ |x|^\delta \left(\cos\frac{\pi}{2}\delta - i\sin\frac{\pi}{2}\delta\right)\left(1+\frac{m\delta}{2}(Y-iX)\frac{1+x^2}{|x|}\right) & if x<0 \end{cases}$$

$$= \quad \begin{cases} |x|^\delta \left[\cos\frac{\pi}{2}\delta - \frac{m\delta}{2}\sin\frac{\pi}{2}\delta(X+\cot\frac{\pi}{2}\delta Y)\frac{1+x^2}{|x|}\right] & if x>0 \\ |x|^\delta \left[\cos\frac{\pi}{2}\delta - \frac{m\delta}{2}\sin\frac{\pi}{2}\delta(X-\cot\frac{\pi}{2}\delta Y)\frac{1+x^2}{|x|}\right] & if x<0. \end{cases}$$

Therefore, for $x\neq 0$, since $\lambda(z_0)$ satisfies (14) and $m\geq 1$, we obtain

$$\Re(p(z_0)+\lambda(z_0)z_0p'(z_0)) \leq |x|^\delta\left[\cos\frac{\pi}{2}\delta - \frac{\delta\eta}{2}\sin\frac{\pi}{2}\delta\left(|x|+\frac{1}{|x|}\right)\right] = f(|x|)$$

Since $f(t)$ with $t=|x|$ attains its maximum value at point

$$t_0 = \frac{\cot\frac{\pi}{2}\delta + \sqrt{\cot^2\frac{\pi}{2}\delta + \eta^2(1-\delta^2)}}{\eta(1+\delta)}.$$

We have

$$\Re(p(z_0)+\lambda(z_0)z_0p'(z_0)) \leq f(|x|) \leq f(t_0) = \beta(\eta,\delta).$$

This is contradiction with our assumption. Hence we must have $p(z)\prec h(z)$. This completes the proof.

Theorem 2.10. *Let $\beta(\eta,\delta)$ be as defined by (15) so that $2\beta(\eta,\delta)+\eta\geq 0$. Let $M(z)=z^n+...$ and $N(z)=z^n+...$ be analytic in U such that for some $\alpha\in C$, N satisfies*

$$\left|\Im\frac{\alpha N(z)}{zN'(z)}\right| \leq \tan\frac{\pi}{2}\delta\left(\Re\frac{\alpha N(z)}{zN'(z)}-\eta\right),$$

Then

$$\Re\left[(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)}\right] > \beta(\eta,\delta),$$

implies

$$\left|\arg\frac{M(z)}{N(z)}\right| < \frac{\pi}{2}\delta.$$

Proof. Consider the function $p(z)=\frac{M(z)}{N(z)}$ and let $\lambda(z)=\frac{\alpha N(z)}{zN'(z)}$. Then by hypothesis,$p$ is analytic and $p(0)=1$ and all conditions of Lemma 2.9 are satisfied.Now it is elementary to show that,

$$(1-\alpha)\frac{M(z)}{N(z)}+\alpha\frac{M'(z)}{N'(z)} = p(z)+\lambda(z)zp'(z),$$

and hence Theorem 2.10 follows from Lemma 2.9.

Taking $M(z) = D^{\nu} f(z)$ and $N(z) = z$ in Theorem 2.10 we have.

Corollary 2.11. *Let $f \in \mathcal{A}$, and let α be complex number satisfies*

$$|\Im\alpha| \leq (\tan \frac{\pi}{2}\delta)\Re\alpha.$$

Then

$$\Re\left[(1-\alpha)\frac{D^{\nu} f(z)}{z} + \alpha(D^{\nu} f(z))'\right] > \beta(\eta, \delta) \quad (0 < \delta \leq 1, \nu \geq 0)$$

implies

$$\left|\arg \frac{D^{\nu} f(z)}{z}\right| < \frac{\pi}{2}\delta,$$

where $\eta = [\Re\alpha - \cot \frac{\pi}{2}\delta|\Im\alpha|]$.

Theorem 2.12. *Let $f \in \mathcal{A}$. If*

$$\Re\left[(1-\lambda)\frac{D^{1+\nu} f(z)}{z} + \lambda(D^{1+\nu} f(z))'\right] > \beta(\eta, \delta) \quad (0 < \delta \leq 1, \nu \geq 0, \lambda \in \mathbb{C})$$

then

$$\arg\left[(1-\lambda)\frac{D^{\nu} f(z)}{z} + \lambda(D^{\nu} f(z))'\right] < \frac{\pi}{2}\delta,$$

where $\beta(\eta, \delta)$ is defined by (15) with $\eta = \frac{1}{1+\nu}$.

Proof. Suppose

$$p(z) = (1-\lambda)\frac{D^{\nu} f(z)}{z} + \lambda(D^{\nu} f(z))'. \tag{17}$$

It is clear that p is analytic and $p(0) = 1$. Differentiating of (17) with respect to z,multiplying by z and using the identity (8) we obtain

$$(1-\lambda)\frac{D^{1+\nu} f(z)}{z} + \lambda(D^{1+\nu} f(z))' = p(z) + \frac{1}{1+\nu} zp'(z). \tag{18}$$

Hence the result follows from(18) and Lemma 2.9.

To prove our next theorem, we shall need the following result, which is due to Miller and Mocanu [3].

Lemma 2.13. *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times U \mapsto \mathbb{C}$ satisfies $\psi(ix, y, z) \notin \Omega$, for all real x, y with $y \leq -\frac{1+x^2}{2}$ and $z \in U$.If the function $zp \in \mathcal{A}$ satisfies $\psi(p(z), zp'(z), z) \in \Omega, z \in U$, then $\Re p(z) > 0$ holds for all $z \in U$.*

Theorem 2.14. *Let $f \in \mathcal{A}$ and $\nu \geq 1$. Also let $\delta \approx 0.638324$ and $\gamma > 0$ be the roots of the equations (respectively)*

$$\delta \tan \frac{\pi}{2}\delta = 1, \tag{19}$$

and

$$\gamma = \nu \tan \frac{\pi}{2}(\delta - \gamma). \tag{20}$$

If $\alpha \geq 1$ and

$$\Re\left((1-\alpha)\frac{D^\nu f(z)}{z} + \alpha(D^\nu f(z))'\right) > \frac{-(1-\frac{1}{\alpha})(2\xi(\alpha)-1)}{1-(1-\frac{1}{\alpha})(2\xi(\alpha)-1)}, \tag{21}$$

then

$$\Re\left(\frac{D^\nu f(z)}{D^{\nu-1} f(z)}\right) > \beta, \tag{22}$$

where $\xi(\alpha) = \int_0^1 \frac{dt}{1+t^\alpha}$ and β is the smallest positive root of the equation

$$\frac{2\sqrt{[\nu(1-\beta)+\frac{1}{2}][\beta\nu + \frac{1}{2} - \frac{\beta}{1-\beta}]}}{|1-\nu+2\beta\nu|} = \tan\frac{\pi}{2}\gamma. \tag{23}$$

Proof. From (21) and using the well-known result of Hallenbeck and Ruscheweyh [1] with identity

$$\frac{D^\nu f(z)}{z} + z\left(\frac{D^\nu f(z)}{z}\right)' = \left(1-\frac{1}{\alpha}\right)\frac{D^\nu f(z)}{z} + \frac{1}{\alpha}\left[\frac{D^\nu f(z)}{z} + \alpha z\left(\frac{D^\nu f(z)}{z}\right)'\right],$$

we observe

$$\Re\left[\frac{D^\nu f(z)}{z} + z\left(\frac{D^\nu f(z)}{z}\right)'\right] > 0. \tag{24}$$

Applying Corollary 2.11 to (20) we get

$$\left|\arg \frac{D^\nu f(z)}{z}\right| < \frac{\pi}{2}\delta,$$

where δ is defined by (19).

Now by using the identity $\frac{D^\nu f(z)}{z} = \frac{\nu-1}{\nu}\frac{D^{\nu-1}}{z} + \frac{1}{\nu}(D^{\nu-1}f(z))'$ and Corollary 2.4 we obtain

$$\left|\arg \frac{D^{\nu-1} f(z)}{z}\right| < \frac{\pi}{2}\gamma,$$

where γ is defined by (20).

Setting $p(z) = \left(\frac{D^\nu f(z)}{D^{\nu-1}f(z)} - \beta\right)\frac{1}{1-\beta}, F(z) = \frac{D^{\nu-1}f(z)}{z}$, and by performing differentiation and some algebraic simplifications, (24) deduces to

$$\Re\psi(p(z), zp'(z), z) > 0$$

where

$$\begin{aligned}\psi(r,s,z) &= F(z)\left(\beta^2\nu + \beta - \beta\nu + r[(1-\beta)(1-\nu) + 2\beta(1-\beta)\nu]\right)\\ &+ F(z)\left(r^2\nu(1-\beta)^2 + (1-\beta)s\right).\end{aligned}$$

Let us now put $F(z) = X + iY$ and apply Lemma 2.13.Then for all x, y real and $z \in U$ we have

$$\begin{aligned}\Re\psi(ix,y,z) &= X\left((\beta^2\nu + \beta - \beta\nu) - \nu(1-\beta)^2x^2 + (1-\beta)y\right)\\ &- Yx[(1-\beta)(1-\nu) + 2\beta(1-\beta)].\end{aligned}$$

From this we observe that

$$\Re\psi(ix,y,z) \le -(ax^2 + bx + c),$$

for all x real,$y \le \frac{-(1+x^2)}{2}$ and $z \in U$, where

$$a = X\left[\nu(1-\beta)^2 + \frac{(1-\beta)}{2}\right], b = Y\left[(1-\beta)(1-\nu) + 2\beta(1-\beta)\nu\right] \quad \textit{and}$$

$$c = X\left[\beta\nu(1-\beta) - \beta + \frac{1-\beta}{2}\right].$$

Therefore $\Re\psi(ix,y,z) \le 0$ if and only if $b^2 \le 4ac$ this indeed equivalent to

$$|\arg F(z)| < \frac{2\sqrt{[\nu(1-\beta) + \frac{1}{2}][\beta\nu + \frac{1}{2} - \frac{\beta}{1-\beta}]}}{|1-\nu+2\beta\nu|} = \tan\frac{\pi}{2}\gamma.$$

Hence if β be the smallest root of the equation (23) then $\Re\psi(ix,y,z) \le 0$ and so by Lemma 2.13 we obtain $\Re p(z) > 0$ which is desired conclution.Therefore the proof is complete.

Corollary 2.15. *Let $f \in \mathcal{A}$ and $\delta \approx 0.638324$ and $\gamma \approx 0.39747$ be the roots of the equations (respectively),*

$$\delta\tan\frac{\pi}{2}\delta = 1 \qquad \textit{and} \qquad \gamma = \tan\frac{\pi}{2}(\delta - \gamma).$$

If

$$\Re(f'(z) + zf''(z)) > 0,$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta,$$

where $\beta \approx 0.46085$ *is the smallest positive root of the equation*

$$\frac{\sqrt{(\frac{3}{2}-\beta)\left(\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{\beta} = \tan\frac{\pi}{2}\gamma.$$

Corollary 2.16. *Let* $f \in \mathcal{A}$ *and* $\delta \approx 0.638324$ *and* $\gamma \approx 0.4864$ *be the smallest roots of the equations (respectively),*

$$\delta \tan\frac{\pi}{2}\delta = 1 \qquad \text{and} \qquad \gamma = 2\tan\frac{\pi}{2}(\delta-\gamma).$$

If

$$\Re(f'(z) + 2zf''(z) + \frac{1}{2}z^2f'''(z)) > 0,$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 2\beta - 1,$$

where $\beta \approx 0.57669$ *is the the smallest positive root of the equation*

$$\frac{2\sqrt{(\frac{5}{2}-2\beta)\left(2\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{4\beta-1} = \tan\frac{\pi}{2}\gamma.$$

*Proof.*Put $\nu = 2$ in Theorem 2.14.

We also note that by using Corollary 2.15 one can get the other new sufficient condition for convexity such as

Corollary 2.17. *Let* $f \in \mathcal{A}$ *and* $\delta \approx 0.638324$ *and* $\gamma \approx 0.39747$ *be the roots of the equations,(respectively)*

$$\delta \tan\frac{\pi}{2}\delta = 1 \qquad \text{and} \qquad \gamma = \tan\frac{\pi}{2}(\delta-\gamma).$$

If

$$\Re(f'(z) + 3zf''(z) + z^2f'''(z)) > 0.$$

Then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta,$$

where $\beta \approx 0.46085$ *is the smallest positive root of the equation*

$$\frac{\sqrt{\left(\frac{3}{2}-\beta\right)\left(\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{\beta}=\tan\frac{\pi}{2}\gamma.$$

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