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NOTE ON THE SOLVABILITY OF A SYSTEM OF EQUATIONS

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Abstract. In this note we formulate sufficient conditions for the solvability of a system of equations in R^d $(d \geq 1)$ using attached polynomial system of equations. The solution of the last system tends to the solution of the original system and the approximation error will be estimated by means of the modulus of smoothness and $K-$functional, respectively.

1. **Introduction**

Let $(X, \|\cdot\|_X)$ be a real or complex normed space and denote by $L(X)$ the space of all continuous linear operators from X to X. For an operator $A \in L(X)$ and an element $y \in X$ let us consider the equation $(I - A)(x) = y$. Approximating the operator A by another operator $\tilde{A} \in L(X)$ and the element y by $\tilde{y} \in X$, we arrive at a new equation $(I - \tilde{A})(\tilde{x}) = \tilde{y}$. This equation usually is easier to solve and it is called the near equation of $(I - A)(x) = y$. The problem to give estimations of the error $\|x - \tilde{x}\|_X$ with the aid of A, $\tilde{A}$, y and $\tilde{y}$ has been studying extensively (see e.g. [6]).

The algorithm described in [4] provides the solutions of the system of equations $f_i(x_1, x_2, \ldots, x_d) = 0$, $i \in \{1, 2, \ldots, d\}$, located in $\Pi_{i=1}^d$ $[0, 1]$, and a polynomial system of equations is used in place of the near equation.

The purpose of this paper is to give sufficient conditions regarding the functions f_i which imply the solvability of the system of equations $f_i(x_1, x_2, \ldots, x_d) = 0$, $i \in \{1, 2, \ldots, d\}$, $d \geq 2$, using different attached polynomial system in comparison

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with [4]. This last system will be given by means of the multivariate Bernstein - Durrmeyer polynomials defined on a simplex. The approximation error will be estimated using a $K-$functional. The case $d = 1$ is treated separately, where the attached equation contains the well - known Bernstein polynomial, and the approximation error is estimated by the Ditzian - Totik modulus of smoothness.

2. **Main results**

For a function $f : [0,1] \to R$ the Bernstein polynomials are defined by

$$(B_n f)(x) \ = \ \sum_{k=0}^{n} p_{n,k}(x)\ f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) \ = \ \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}, \quad x \in [0,1], \quad n \geq 1.$$

Let us consider the equation

$$f(x) \ = \ 0 \tag{1}$$

and let

$$(B_n f)(x) \ = \ 0 \tag{2}$$

be the attached equation to (1). Our first result is:

Theorem 1. *Let $f : [0,1] \to R$ be a continuously differentiable function such that $f(0) \cdot f(1) < 0$ and there exists $q > 0$ with the property $|f'(x)| \geq q$ for all $x \in [0,1]$. If $y \in [0,1]$ is a solution of the equation (1) then there exists a sequence $(x_n)_{n\geq 1}$ such that x_n is a solution of (2) for all $n \geq 1$ and $\lim_{n\to\infty} x_n = y$. Moreover, we have the estimations*

$$|x_n - y| \ \leq \ \frac{1}{q}\ (10 + 4\sqrt{3})\ \omega_\varphi^2\left(f, n^{-1/2}\right), \qquad n \geq 1,$$

where

$$\omega_\varphi^2\left(f, n^{-1/2}\right) \ = \ \sup_{0<t\leq n^{-1/2}}\ \sup_{x\in[0,1]}\ |f(x+t\varphi(x)) - 2f(x) + f(x-t\varphi(x))|,$$

$\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ is the Ditzian - Totik modulus of smoothness. Furthermore, if $f \in C^2[0,1]$ then

$$\lim_{n\to\infty} n(B_n f)(y) \;=\; \frac{1}{2}\, y(1-y)f''(y)$$

and

$$|x_n - y| \;\le\; \frac{1}{qn}\, \left(\frac{5}{2} + \sqrt{3}\right)\, \|f''\|_\infty,$$

where $n \ge 1$ and $\|\cdot\|_\infty$ is the sup - norm on $[0,1]$.

Proof. The hypotheses $f(0)\cdot f(1) < 0$ and $|f'(x)| \ge q$, $x \in [0,1]$ imply that y is the unique solution of (1). Furthermore, $(B_n f)(0) = f(0)$ and $(B_n f)(1) = f(1)$. So, by $f(0)\cdot f(1) < 0$ we obtain $(B_n f)(0)\cdot(B_n f)(1) < 0$, which implies the existence of a solution x_n of the equation (2) for all $n \ge 1$. On the other hand, in view of Lagrange's mean - value theorem there exists z_n between y and x_n such that

$$f(x_n) = f(x_n) - f(y) = f'(z_n)\cdot(x_n - y).$$

Hence, by $|f'(x)| \ge q$, $x \in [0,1]$ we get

$$\begin{aligned} |x_n - y| &\le \frac{1}{q}\cdot|f(x_n)| = \frac{1}{q}\cdot|f(x_n) - (B_n f)(x_n)| \\ &\le \frac{1}{q}\cdot \max\{|f(x) - (B_n f)(x)| \;:\; x \in [0,1]\} \\ &= \frac{1}{q}\cdot\|f - B_n f\|_\infty \to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{3}$$

This means that $\lim_{n\to\infty} x_n = y$. Using (3) and [5, p. 452, Corollary 11] we have

$$|x_n - y| \;\le\; \frac{1}{q}\, (10 + 4\sqrt{3})\, \omega_\varphi^2\left(f, n^{-1/2}\right) \tag{4}$$

If $f \in C^2[0,1]$ then, in view of Voronovskaja theorem [3, p. 307, Theorem 3.1] and $f(y) = 0$,

$$\lim_{n\to\infty} n(B_n f)(y) \;=\; \frac{1}{2}\, y(1-y)f''(y).$$

Using the definition of $\omega_\varphi^2\left(f, n^{-1/2}\right)$ and [7, p. 47, (2)], we obtain

$$f(x + t\varphi(x)) - 2f(x) + f(x - t\varphi(x)) \;=\; \int_{-\frac{t}{2}\varphi(x)}^{\frac{t}{2}\varphi(x)} \int_{-\frac{t}{2}\varphi(x)}^{\frac{t}{2}\varphi(x)} f''(x + u_1 + u_2)\; du_1 du_2.$$

Hence

$$\omega_\varphi^2\left(f, n^{-1/2}\right) \;\le\; \|f''\|_\infty \cdot \sup_{0<t\le n^{-1/2}} \; \sup_{x\in[0,1]} t^2\varphi^2(x) \;\le\; \frac{1}{4n}.$$

By (4) we arrive at the estimation

$$|x_n - y| \;\le\; \frac{1}{qn}\cdot\left(\frac{5}{2}+\sqrt{3}\right)\cdot\|f''\|_\infty,$$

which completes the proof.

Using same ideas it can be proved the following:

Corollary 1. *Let $f:[0,1]\to R$ be a continuous function such that $f(0)\cdot f(1) < 0$ and there exists $q>0$ with the property $q|x-x'| \le |f(x)-f(x')|$ for all $x, x' \in [0,1]$. If $y\in[0,1]$ is a solution of the equation (1) then there exists a sequence $(x_n)_{n\ge 1}$ such that x_n is a solution of (2) for all $n\ge 1$ and*

$$|x_n - y| \;\le\; \frac{1}{q}\left(10+4\sqrt{3}\right)\,\omega_\varphi^2\left(f, n^{-1/2}\right).$$

Remark 1. *A solution x_n of the equation $(B_n f)(x) = 0$, $n\ge 1$, can be obtained by Bairstow's method* [9, pp. 301 - 303].

In what follows we consider the multivariate Bernstein - Durrmeyer polynomials introduced by Derriennic [2] as

$$(M_n f)(x) \;=\; \frac{(n+d)!}{n!}\sum_{(\beta/n)\in T} P_{n,\beta}(x)\int_T P_{n,\beta}(u) f(u)\,du,$$

where $x, u\in R^d$, $x=(x_1,\dots,x_d)$, $u=(u_1,\dots,u_d)$, $\beta=(k_1,\dots,k_d)$ with k_i integers, and $T=\{u: 0\le u_i, \sum_{i=1}^d u_i \le 1\}$. Furthermore, $P_{n,\beta}(u)$ is given by

$$P_{n,\beta}(u) \;=\; \frac{n!}{\beta!(n-|\beta|)!}\,u^\beta(1-|u|)^{n-|\beta|},$$

where $\beta! = k_1!\dots k_d!$, $u^\beta = u_1^{k_1}\dots u_d^{k_d}$ ($u_i^{k_i}=1$ if $k_i=u_i=0$), $|u|=\sum_{i=1}^d u_i$ and $|\beta| = \sum_{i=1}^d k_i$. We define, by virtue of [1, p. 112, (2.9)],

$$P(D) \;=\; \sum_{i=1}^d \frac{\partial}{\partial x_i}\, x_i(1-|x|)\,\frac{\partial}{\partial x_i} \;+\; \sum_{i<j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right) x_i x_j \left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right).$$

Now we consider the following system of equations:

$$f_i(x) \;=\; 0, \qquad i\in\{1,2,\dots,d\}, \tag{5}$$

where $f_i : T \to R$, and let

$$(M_n f_i)(x) \;=\; 0, \qquad i \in \{1, 2, \ldots, d\} \tag{6}$$

be the attached system of equations to (5). We denote by $\|\cdot\|$ a norm on R^d.

Theorem 2. *Let $(f_1, \ldots, f_d) : T \to R^d$ be a continuous function. If $y \in T$ is a solution of (5), $(x^n)_{n\geq 1}$, $x^n = (x_1^n, \ldots, x_d^n)$ is a solution of (6) for all $n \geq 1$ and there exist $q > 0$ and $i_0 \in \{1, 2, \ldots, d\}$ with the property $q\|x - x'\| \leq |f_{i_0}(x) - f_{i_0}(x')|$ for all $x, x' \in T$ then $\lim_{n\to\infty} \|x^n - y\| = 0$. Moreover, we have the estimation*

$$\|x^n - y\| \;\leq\; \frac{2}{q}\, K\left(f_{i_0}, n^{-1}\right), \qquad n \geq 1,$$

where

$$K\left(f_{i_0}, n^{-1}\right) \;=\; \inf\,\{\; \|f_{i_0} - g\|_\infty + n^{-1}\; \|P(D)g\|_\infty \;:\; g \in C^2(T)\;\}$$

and $\|\cdot\|_\infty$ is the sup - norm on T. If $f \in C^2(T)$ then

$$\lim_{n\to\infty}\; n(M_n f_{i_0})(y) = P(D) f_{i_0}(y).$$

Proof. We have

$$\begin{aligned}
\|x^n - y\| \;&\leq\; \frac{1}{q}\cdot |f_{i_0}(x^n) - f_{i_0}(y)| = \frac{1}{q}\cdot |f_{i_0}(x^n)| \\
&=\; \frac{1}{q}\cdot |f_{i_0}(x^n) - (M_n f_{i_0})(x^n)| \\
&\leq\; \frac{1}{q}\cdot \|M_n f_{i_0} - f_{i_0}\|_\infty \leq \frac{2}{q}\cdot K\left(f_{i_0}, n^{-1}\right), \quad n \geq 1,
\end{aligned}$$

in view of [1, p. 115, (3.2)]. Hence $\lim_{n\to\infty} \|x^n - y\| = 0$.

If $f \in C^2(T)$, then, by [1, p. 112, Lemma 2.1] and $f_{i_0}(y) = 0$ we obtain

$$\lim_{n\to\infty}\; n(M_n f_{i_0})(y) \;=\; P(D) f_{i_0}(y),$$

which was to be proved.

Remark 2. *Let $q > 0$ and $f = (f_1, \ldots, f_d) : R^d \to R^d$ be a differentiable function with $\|f'(x)\|_* = \sup\{\|f'(x)(z)\| \;:\; \|z\| \leq 1\} \geq q$ for all $x \in R^d$. Then the condition $\|f(x) - f(x')\| \geq q\|x - x'\|$ for all $x, x' \in R^d$ is not necessarily true for $d \geq 2$ (see* [8, p. 81, 3.23]).

Remark 3. Following [4] we have: *the system of equations (6) can be transformed into an equivalent "triangular" polynomial system using the Gröbner basis algorithm. So the solvability of the last system of equations can be traced back to the solvability of a polynomial equation with one unknown. To solve this equation we apply again Bairstow's method on* $[0, 1]$. *After that we generate all solutions of the "triangular" polynomial system located in T. Thus we arrive at the solutions* x^n *of (6),* $n \geq 1$. *It may happen that the polynomial equation with one unknown has not solutions in* $[0, 1]$. *In this case the polynomial system of equation has not solution either and the polynomial system must be rephrased.*

Remark 4. *In* [4] *another attached system of equation is given, namely*

$$(\tilde{B}_n f_i)(x) = 0, \qquad i \in \{1, 2, \ldots, d\},$$

where $x = (x_1, \ldots, x_d) \in D$, $D = \prod_{i=1}^{d} [0, 1]$, $f_i : D \to R$ *and*

$$(\tilde{B}_n f_i)(x) \ = \ \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_d=1}^{n} f_i\left(\frac{i_1}{n}, \ldots, \frac{i_d}{n}\right) \prod_{j=1}^{d} \left(\begin{array}{c} n \\ i_j \end{array} \right) (x_j)^{i_j} \cdot (1 - x_j)^{n-i_j}.$$

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