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THE BETA APPROXIMATING OPERATORS OF SECOND KIND

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Abstract. We shall define a general linear transform from which we obtain as particular case the beta second kind transform:

$$T_{p,q}f = \frac{1}{B(p,q)}\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}f(u^a)du \qquad (*)$$

We consider here only the particular case $a = 1$.

We obtain several positive linear operators as a particular case of this beta second kind transform. We apply the transform $(*)$ to Baskakov's operator and Bleimann, Butzer and Hahn operator respectively and we obtain new generalization of these operators.

1. **Introduction**

Many authors introduced and studied positive linear operators, using Euler's beta function of second kind: [1], [2], [5], [6], [7], [9].

Euler's beta function of second kind is defined for $p > 0$, $q > 0$ by the following formula

$$B(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}du \qquad (1.1)$$

The beta transform of the function f is defined by the following formula

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)}\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}f(u)du$$

We shall define a more general linear transform from which we obtain as particular case the beta second-kind transform.

For $a, b \in \mathbb{R}$ we define the (a,b)-beta transform of a function f

$$\mathcal{B}_{p,q}^{(a,b)}f = \frac{1}{B(p,q)}\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}f\left(\frac{u^a}{(1+u)^{a+b}}\right)du \qquad (1.2)$$

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where $B(\cdot,\cdot)$ is the beta function (1.1) and f is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f| < \infty$.

2. **The beta second-kind transform. Case** $a = 1$

Let us denote by $M[0,\infty)$ the linear space of functions defined for $t \geq 0$, bounded and Lebesgue measurable in each interval $[c,d]$, where $0 < c < d < \infty$.

If we consider in (1.2) $a + b = 0$ we obtain the second-kind transform of function $f \in M[0,\infty)$

$$T_{p,q}^{(a)} f = \mathcal{B}_{p,q}^{(a,-a)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u^a) du \tag{2.1}$$

such that $T_{p,q}^{(a)}|f| < \infty$. Clearly $T_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here only the particular case $a = 1$ (see also [9])

$$T_{p,q} f = T_{p,q}^{(1)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f(u) du \tag{2.2}$$

for $f \in M[0,\infty)$ such that $T_{p,q}|f| < \infty$.

Remark. If $a = -1$ we obtain

$$T_{p,q}^{(-1)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{1}{u}\right) du$$

Denoting $u = v^{-1}$ we can write

$$T_{p,q}^{(-1)} f = \frac{1}{B(p,q)} \int_0^\infty \frac{\left(\frac{1}{v}\right)^{p-1}}{\left(1+\frac{1}{v}\right)^{p+q}} f(v) \frac{1}{v^2} dv =$$

$$= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{p+q}}{v^{p+1}(1+v)^{p+q}} f(v) dv =$$

$$= \frac{1}{B(p,q)} \int_0^\infty \frac{v^{q-1}}{(v+1)^{p+q}} f(v) dv = T_{p,q}^{(1)} f.$$

That is $T_{p,q}^{(-1)} f = T_{p,q}^{(1)} f = T_{p,q} f$.

Lemma 2.1. [9] *The moment of order k $(1 \leq k < q)$ of the functional $T_{p,q}$ has the following value*

$$T_{p,q} e_k = \frac{p(p+1)\dots(p+k-1)}{(q-1)\dots(q-k)}, \quad 1 \leq k < q \tag{2.3}$$

We impose that $T_{p,q}e_1 = e_1$, that is $p = (q-1)x$, $q > 1$ and we obtain

$$(T_q f)(x) = \frac{1}{B((q-1)x, q)} \int_0^{\infty} \frac{u^{(q-1)x-1}}{(1+u)^{(q-1)(x+1)+1}} f(u)du \tag{2.4}$$

Lemma 2.2. *One has*

$$(T_q e_2)(x) = x^2 + \frac{x(x+1)}{q-2}, \quad q > 2$$

$$T_q((t-x)^2; x) = \frac{x(x+1)}{q-2}, \quad q > 2. \tag{2.5}$$

Proof. It is obtained from Lemma 2.1 for $p = (q-1)x$. □

Particular cases

a) If in (2.4) we choose $q = 1 + \frac{1}{\alpha}$, $\alpha \in (0,1)$, then we give the positive linear operator L_α defined for $\alpha \in (0,1)$ and $x \geq 0$:

$$(L_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha} + 1\right)} \int_0^{\infty} \frac{u^{\frac{x}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha}+1}} f(u)du \tag{2.6}$$

considered in [9] (see also [1], [2], [7]).

Lemma 2.3. *One has*

$$(L_\alpha e_2)(x) = x^2 + \frac{\alpha}{1-\alpha} x(1+x).$$

$$L_\alpha((t-x)^2; x) = \frac{\alpha}{1-\alpha} x(1+x).$$

Proof. We take $q = (\alpha+1)/\alpha$ in (2.5). □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$L_{1/n}((t-x)^2; x) = \frac{x(1+x)}{n-1}.$$

b) If we choose in (2.4) $q = \frac{1}{\alpha(1+x)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \frac{1}{\alpha} - 1\right)$, we obtain the beta type operator H_α, given by

$$(H_\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha(1+x)}, \frac{1}{\alpha(1+x)} + 1\right)} \int_0^{\infty} \frac{u^{\frac{x}{\alpha(1+x)}-1}}{(1+u)^{\frac{1}{\alpha}+1}} f(u)du \tag{2.7}$$

where $f \in M[0,\infty)$ such that $H_\alpha|f| < \infty$, considered by J. Adell [2].

Lemma 2.4. *One has*

$$(H_\alpha e_2)(x) = x^2 + \frac{\alpha x(1+x)^2}{1-\alpha(x+1)}$$

$$H_\alpha((t-x)^2;x) = \frac{\alpha x(1+x)^2}{1-\alpha(x+1)}$$

Proof. We take $q = \dfrac{1}{\alpha(x+1)} + 1$ in Lemma 2.2. □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$H_{1/n}((t-x)^2;x) = \frac{x(1+x)^2}{n-1-x}.$$

c) If we put in (2.4) $q = 1 + \dfrac{1}{\alpha x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain the positive linear operator M_α, given by

$$(M_\alpha f)(x) = \frac{1}{B\left(\frac{1}{\alpha}, \frac{1}{\alpha x}+1\right)} \int_0^\infty \frac{u^{\frac{1}{\alpha}-1}}{(1+u)^{\frac{1+x}{\alpha x}+1}} f(u)du \qquad (2.8)$$

where $f \in M(0,\infty)$ such that $M_\alpha|f| < \infty$.

Lemma 2.5. *One has*

$$(M_\alpha e_2)(x) = x^2 + \frac{\alpha x^2(x+1)}{1-\alpha x}$$

$$M_\alpha((t-x)^2;x) = \frac{\alpha x^2(x+1)}{1-\alpha x}$$

Proof. The above identities are implied by Lemma 2.2. □

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$M_{1/n}((t-x)^2;x) = \frac{x^2(x+1)}{n-x}.$$

3. **Generalized Baskakov operator**

Let be $\overline{B}_n$ the Baskakov operator [3]

$$(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right) \tag{3.1}$$

Now let us apply the transform $T_{p,q}$ (2.2) to Baskakov's operator (3.1) and we obtain (see [9])

Theorem 3.1. *The $T_{p,q}$ transform of $\overline{B}_n f$ can be expressed by the following form*

$$\overline{T}_n^{(p,q)} f = T_{p,q}(\overline{B}_n f) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{n+k}} f\left(\frac{k}{n}\right) \tag{3.2}$$

where $(a)_m := a(a+1)\dots(a+m-1)$.

Proof. $\overline{T}_n^{(p,q)} f = T_{p,q}(\overline{B}_n f) =$

$$= \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{u^k}{(1+u)^{n+k}} f\left(\frac{k}{n}\right) du =$$

$$= \frac{1}{B(p,q)} \sum_{k=0}^{\infty} \binom{n+k-1}{k} f\left(\frac{k}{n}\right) \int_0^{\infty} \frac{u^{p+k-1}}{(1+u)^{p+q+n+k}} du =$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{B(p+k,q+n)}{B(p,q)} f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{(p)_k (q)_n}{(p+q)_{k+n}} f\left(\frac{k}{n}\right). \square$$

Theorem 3.2. *One has*

$$\overline{T}_n^{(p,q)} e_1 = T_{p,q}(\overline{B}_n e_1) = \frac{p}{q-1} \tag{3.3}$$

$$\overline{T}_n^{(p,q)} e_2 = T_{p,q}(\overline{B}_n e_2) = \frac{p(p+1)}{(q-2)(q-1)} + \frac{1}{n} \frac{p(p+q-1)}{(q-2)(q-1)}.$$

Proof. $\overline{T}_n^{(p,q)} e_1 = \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} u du$

$$\frac{1}{B(p,q)} \int_0^{\infty} \frac{u^p}{(1+u)^{p+q}} du = \frac{B(p+1,q-1)}{B(p,q)} = \frac{p}{q-1}.$$

$$\overline{T}_n^{(p,q)} e_2 = \frac{1}{B(p,q)} \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} \left(u^2 + \frac{u(u+1)}{n}\right) du =$$

$$= \frac{1}{B(p,q)} \left(\int_0^{\infty} \frac{u^{p+1}}{(1+u)^{p+q}} du + \frac{1}{n} \int_0^{\infty} \frac{u^p}{(1+u)^{p+q-1}} du \right) =$$

$$= \frac{1}{B(p,q)}\left(B(p+2,q-2)+\frac{1}{n}B(p+1,q-2)\right)=$$

$$= \frac{B(p+2,q-2)}{B(p,q)}+\frac{1}{n}\frac{B(p+1,q-2)}{B(p,q)} = \frac{p(p+1)}{(q-2)(q-1)}+\frac{1}{n}\frac{p(p+q-1)}{(q-2)(q-1)}. \square$$

We impose that $\overline{T}_n^{(p,q)}e_1 = e_1$, that is $p=(q-1)x$, $x>0$, $q>2$. We obtain from Theorem 3.1 and Theorem 3.2

Corollary 3.3. *One has*

$$\overline{T}_n^{(q)}f = T_q(\overline{B}_n f) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{((q-1)x)_k(q)_n}{((q-1)x+q)_{n+k}}f\left(\frac{k}{n}\right) \tag{3.4}$$

Corollary 3.4. *One has*

$$(\overline{T}_n^{(q)}e_1)(x)=x, \quad (\overline{T}_n^{(q)}e_2)(x)=x^2+\frac{x(1+x)}{q-2}\left(1+\frac{q-1}{n}\right)$$

$$\overline{T}_n^{(q)}((t-x)^2;x)=\frac{x(1+x)}{q-2}\left(1+\frac{q-1}{n}\right) \tag{3.5}$$

Proof. Choosing $p=(q-1)x$ in Theorem 3.2, the conclusion follows. $\square$

Particular cases

a) If we put in (3.4) $q=\frac{1}{\alpha}+1$, $\alpha\in(0,1)$, we obtain the operator considered by D. D. Stancu [8], as a generalization of the Baskakov operator

$$(\overline{L}_n^{(\alpha)}f)(x) = L_\alpha(\overline{B}_n f)(x)=\sum_{k=0}^{\infty}\bar{l}_{n,k}(x,\alpha)f\left(\frac{k}{n}\right) \tag{3.6}$$

where

$$\bar{l}_{n,k}(x,\alpha)=\binom{n+k-1}{k}\frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)(1+2\alpha)\dots(1+n\alpha)}{(1+x+\alpha)(1+x+2\alpha)\dots(1+x+(n+k)\alpha)}$$

Corollary 3.5. *One has*

$$\overline{L}_n^{(\alpha)}((t-x)^2;x)=\frac{\alpha x(1+x)}{1-\alpha}\left(1+\frac{1}{n\alpha}\right)$$

For $\alpha=1/n$, $n\in\mathbb{N}$, we obtain

$$\overline{L}_n((t-x)^2;x)=\frac{2x(1+x)}{n-1}.$$

b) For $q = \dfrac{1}{\alpha(x+1)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$, we obtain by (3.4) a new generalization of the Baskakov operator

$$(\overline{H}_n^{(\alpha)} f)(x) = H_\alpha(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \overline{h}_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{3.7}$$

where

$$\overline{h}_{n,k}(x,\alpha) =$$

$$= \binom{n+k-1}{k} \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(x+1))(1+\alpha(1+x))\dots(1+n\alpha(1+x))}{(1+x)^{n+k}(1+\alpha)\dots(1+(n+k)\alpha)}$$

Corollary 3.6. *One has*

$$\overline{H}_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha x(1+x)^2}{1-\alpha(1+x)}\left(1 + \frac{1}{\alpha n(1+x)}\right)$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{H}_n((t-x)^2;x) = \frac{x(x+1)(x+2)}{n-1-x}.$$

c) If we put in (3.4) $q = 1 + \dfrac{1}{\alpha x}$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain a new generalization of the Baskakov operator

$$(\overline{M}_n^{(\alpha)} f)(x) = M_\alpha(\overline{B}_n f)(x) = \sum_{k=0}^{\infty} \overline{m}_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{3.8}$$

where

$$\overline{m}_{n,k}(x,\alpha) = \binom{n+k-1}{k} \frac{(1+\alpha)\dots(1+(k-1)\alpha)(1+\alpha x)\dots(1+n\alpha x)}{(x+1+\alpha x)\dots(x+1+(n+k)\alpha x)} x^k$$

Corollary 3.7. *One has*

$$\overline{M}_n^{(\alpha)}((t-x)^2;x) = \frac{\alpha x^2(x+1)}{1-\alpha x}\left(1 + \frac{1}{\alpha n x}\right)$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{M}_n((t-x)^2;x) = \frac{x(x+1)^2}{n-x}.$$

4. **Generalized Bleimann, Butzer, Hahn operator**

Let be $\widetilde{B}_n$ the Bleimann, Butzer, Hahn operator [4]

$$(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right) \tag{4.1}$$

Now let us apply the transform $T_{p,q}$ (2.2) to Bleimann, Butzer, Hahn's operator (4.1) and we obtain

Theorem 4.1. *The $T_{p,q}$ transform of $\widetilde{B}_n f$ can be expressed by the following form*

$$\widetilde{T}_n^{(p,q)} f = T_{p,q}(\widetilde{B}_n f) = \sum_{k=0}^{n} \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} f\left(\frac{k}{n-k+1}\right) \tag{4.2}$$

Proof.

$$\widetilde{T}_n^{(p,q)} f = T_{p,q}(\widetilde{B}_n f) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} \sum_{k=0}^{n} \binom{n}{k} \frac{u^k}{(1+u)^n} f\left(\frac{k}{n-k+1}\right) =$$

$$= \frac{1}{B(p,q)} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n-k+1}\right) \int_0^\infty \frac{u^{p+k-1}}{(1+u)^{p+q+n}} du =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{B(p+k, q+n-k)}{B(p,q)} f\left(\frac{k}{n-k+1}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} f\left(\frac{k}{n-k+1}\right). \ \square$$

Particular cases

a) If we put in (4.2), $p = \dfrac{x}{\alpha}$, $q = \dfrac{1}{\alpha} + 1$, $\alpha \in (0,1)$, $x \geq 0$, we obtain the operator introduced by J. Adell [2] as a generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{L}_n^{(\alpha)} f)(x) = L_\alpha(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \widetilde{l}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.3}$$

where

$$\widetilde{l}_{n,k}(x,\alpha) = \binom{n}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1+\alpha)\dots(1+(n-k)\alpha)}{(x+1+\alpha)\dots(x+1+n\alpha)}$$

b) For $p = \dfrac{x}{\alpha(1+x)}$ and $q = \dfrac{1}{\alpha(1+x)} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha} - 1\right)$, we obtain by (4.2) a new generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{H}_n^{(\alpha)} f)(x) == H_\alpha(\widetilde{B}_n f)(x) \sum_{k=0}^{n} \widetilde{h}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.4}$$

where

$$\widetilde{h}_{n,k}(x,\alpha) =$$

$$= \binom{n}{k} \frac{x(x+\alpha(1+x))\dots(x+(k-1)\alpha(1+x))(1+\alpha(1+x))\dots(1+(n-k)\alpha(1+x))}{(1+x)^n(1+\alpha)(1+2\alpha)\dots(1+n\alpha)}.$$

c) If we put in (4.2) $p = \dfrac{1}{\alpha}$, $q = \dfrac{1}{\alpha x} + 1$, $\alpha \in (0,1)$, $x \in \left(0, \dfrac{1}{\alpha}\right)$, we obtain a new generalization of the Bleimann, Butzer, Hahn operator

$$(\widetilde{M}_n^{(\alpha)} f)(x) = M_\alpha(\widetilde{B}_n f)(x) = \sum_{k=0}^{n} \widetilde{m}_{n,k}(x,\alpha) f\left(\frac{k}{n-k+1}\right) \tag{4.5}$$

where

$$\widetilde{m}_{n,k}(x,\alpha) = \binom{n}{k} \frac{(1+\alpha)\dots(1+k\alpha)(1+\alpha x)\dots(1+(n-k-1)\alpha x)}{(x+1+\alpha x)\dots(x+1+n\alpha x)} x^k.$$

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