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BOOLE LATTICES OF IDEMPOTENTS IN A RING

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Abstract. In this paper, we will show that in a ring R, there exist maximal subsets of commuting idempotents. On these maximal subsets, one can define Boole lattice structures which induce Boole rings which usually are not subrings of R. If R is a Boole ring, we obtain the Stone's Theorem.

Let V be a linear space over the skewfield K and $EndV$ the set of endomorphisms of V.

If $f, g \in EndV$ then the functions $f + g : V \to V$ and $g \circ f : V \to V$ defined by:

$$(f + g)(x) = f(x) + g(x) \text{ and } (f \circ g)(x) = f(g(x))$$

are endomorphisms of V, that is $f + g, f \circ g \in EndV$.

The set $EndV$ is a ring with respect to the operations defined above. This ring is not commutative if $dimV \geq 2$. An endomorphism f of V is called projector of V if $f^2 = f$.

Starting from the papers of W.J. Gordon [3] and W.J. Gordon and E.W.Cheney [4], F.J. Delvos and W. Schempp are presenting in their book [1] the construction of lattices of projectors from $EndV$ which are commutative. They use these lattices in the approximation and interpolation theory.

In this paper, we will associate Boole lattices to a ring (associative) with the unit R. An element $a \in R$ with the property $a^2 = a$ is called idempotent. Thus, the projectors of V correspond to the idempotent elements of the ring $EndV$.

If R is a Boole ring, i.e. every element of $a \in R$ is idempotent, then these lattices coincide with the Boole lattice associated to R, according to Stone's Theorem which establishes a bijection between Boole lattices and Boole rings. Note that every Boole ring R is commutative and $2a = 0$ for $\forall a \in R$.

Let $I(R) = \{a \in R | a^2 = a\}$, $\mathcal{P}(I(R))$ the set of subsets of $I(R)$ and

$$\mathcal{P}' = \{X \in \mathcal{P}(I(R)) \,|\, \forall a, b \in X; ab = ba\}.$$

Remarks. a) We have $I(R) = R$ if and only if R is a Boole ring. In this case, $\mathcal{P}' = \mathcal{P}(R)$ and R is the only maximal element of $\mathcal{P}'$.

b) We have $\{0, 1\} \in \mathcal{P}'$.

Theorem 1. 1) For every $X \in \mathcal{P}'$ there exists a maximal element Y in $(\mathcal{P}', \subseteq)$ such that $X \subseteq Y$.

2) If Y is a maximal element of $\mathcal{P}'$ then $0, 1 \in Y$.

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Proof. 1) If $\mathcal{P}''$ is a non empty chain from $\mathcal{C} = \{X' \in \mathcal{P}' | X \subseteq X'\}$ then

$$\bigcup_{X' \in \mathcal{P}''} X' \in \mathcal{C}$$

Thus, according to Zorn's lemma there exist maximal elements in $\mathcal{C}$.

2) The elements 0 and 1 are idempotent and they commute with every $y \in Y$. So $Y \cup \{0, 1\} \in \mathcal{P}'$ and using the maximality of Y in $\mathcal{P}'$ it results that $0, 1 \in Y$.

Theorem 2. If Y is a maximal element in $(\mathcal{P}', \subseteq)$ then:

i) Y is stable with respect to the multiplication in R, i.e.

$$x, y \in Y \Rightarrow xy \in Y$$

ii) The relation "$\leq$" defined on Y by

$$x \leq y \Leftrightarrow xy = x$$

is an ordering relation and 0 respectively 1 is the least respectively the greatest element in $(Y, \leq)$.

iii) The ordered set $(Y, \leq)$ is a Boole lattice. In this lattice we have

$$x \wedge y = xy,\ x \vee y = x + y - xy \text{ and } x' = 1 - x \tag{1}$$

where $x \wedge y = inf\,(x, y)$, $x \vee y = sup\,(x, y)$ and x' is the complement of x.

Proof. i) From $x, y \in Y$ it results that x, y are idempotents which commute, which implies that

$$(xy)^2 = xyxy = x^2y^2 = xy$$

so xy is idempotent. Since the elements of Y are commuting, it results that for $\forall z \in Y$ we have

$$(xy)\,z = x\,(yz) = x\,(zy) = (xz)\,y = (zx)\,y = z\,(xy)\,,$$

so xy commutes with every element of Y.

This means that $Y \cup \{xy\} \in \mathcal{P}'$, which together with the maximality of Y in $(\mathcal{P}', \subseteq)$, implies that $xy \in Y$.

ii) Since the elements of Y are idempotents, it results that for $\forall x \in X$ we have

$$x^2 = x,$$

so $x \leq x$. Thus the "$\leq$" relation is reflexive. If $x, y, z \in Y$ then

$$x \leq y \text{ and } y \leq z \Rightarrow xy = x \text{ and } yz = y$$

Using the fact that y is idempotent we deduce that

$$(xy)\,(yz) = xy \Rightarrow xyz = xy \Rightarrow xz = x \Rightarrow x \leq z$$

So the "$\leq$" relation is also transitive.

If $x, y \in Y$ then using the fact that the elements of Y are commuting, we have:

$$x \leq y \text{ and } y \leq x \Rightarrow xy = x \text{ and } yx = y \Rightarrow x = y$$

which shows that the relation "$\leq$" is antisymmetric.

We have proved that "$\leq$" is an order relation on Y. For $\forall x \in Y$, from

$$0x = 0 \text{ and } x1 = x$$

it results that $0 \leq x$ and $x \leq 1$.

So, 0 and 1 are the least respectively the greatest element in $(Y, \leq)$.

iii) For $\forall x, y \in Y$ we have

$$(xy)\,x = x^2 y = xy, (xy)\,y = xy^2 = xy$$

and

$$(x + y - xy)\,x = x^2 + yx - xyx = x + yx - yx^2 = x + yx - yx = x,$$

$$(x + y - xy)\,y = xy + y^2 - xy^2 = xy + y - xy = y$$

which shows that $xy \leq x$, $xy \leq y$ and $x \leq x + y - xy$ and $y \leq x + y - xy$, so xy is a lower bound and $x + y - xy$ is a upper bound of x and y.

If $z \in Y$ then

$$z \leq x, z \leq y \Rightarrow zx = z, zy = z \Rightarrow z^2\,(xy) = z^2 \Rightarrow z\,(xy) = z \Rightarrow z \leq xy$$

and

$$x \leq z, y \leq z \Rightarrow xz = x, yz = y$$

from which it results

$$z\,(x + y - xy) = zx + zy - zxy = x + y - xy$$

hence $z \leq x + y - xy$.

So xy is the greatest lower bound and $x + y - xy$ is the least upper bound of x and y.

So

$$x \wedge y = xy \text{ and } x \vee y = x + y - xy$$

Thus we have proved that $(Y, \leq)$ is a lattice.

Now we will show that this lattice is also distributive. If $x, y, z \in Y$ then

$$x \wedge (y \vee z) = x\,(y + z - yz) = xy + xz - xyz,$$

$$(x \wedge y) \vee (x \wedge z) = (xy) \vee (xz) = xy + xz - xyxz = xy + xz - xyz$$

so

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2}$$

Here we notice that the identity (2) is true in a lattice if and only if the following identity is also true:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Thus $(Y, \leq)$ is a distributive lattice having 0 the least element and 1 the greatest element.

For $\forall x \in Y$ we have

$$x \wedge (1 - x) = x - x^2 = x - x = 0,$$

$$x \vee (1 - x) = x + 1 - x - x\,(1 - x) = 1 - x + x = 1$$

which shows that $x' = 1 - x$ is the complement of x.

So we proved that $(Y, \leq)$ is a Boole lattice.

Corollary. If $X \in \mathcal{P}'$ then the relation " $\leq$ " defined on X

$$x \leq y \Leftrightarrow xy = x$$

is an order relation.

Theorem 3. If Y is a maximal element in $(\mathcal{P}', \subseteq)$ then $\oplus$ defined by

$$x \oplus y = x + y - 2xy$$

is an operation on Y and Y is a Boolean ring with respect to $\oplus$ and the multiplication induced by the multiplication in R.

Proof. Applying Stone's Theorem to the Boole lattice $(Y, \vee, \wedge)$ it follows that the equalities

$$\begin{aligned} x * y &= (x \wedge y') \vee (x' \wedge y) \\ xy &= x \wedge y \end{aligned} \tag{3}$$

are defining operations in Y and $(Y, *, \cdot)$ is a Boole ring.

From (1) and (3) it results :

$$\begin{aligned} x * y &= [x(1-y)] \vee [(1-x)y] = x(1-y) + (1-x)y - x(1-y)(1-x)y = \\ &= x - xy + y - xy - x(1-x-y+yx)y = \\ &= x + y - xy - xy - xy + x^2y + xy^2 - xyxy = \\ &= x + y - xy - xy - xy + xy + xy - xy = x + y - 2xy = x \oplus y \end{aligned}$$

Corollary. a) If Y is a maximal element in $\mathcal{P}'$ then the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if

$$2x = 0 \text{ for } \forall x \in Y \tag{4}$$

We know that the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if

$$x \oplus y = x + y \text{ for } \forall x, y \in Y \tag{5}$$

and

$$(5) \Leftrightarrow 2xy = 0; \forall x, y \in Y \Leftrightarrow 2x = 0, \forall x \in Y$$

The last equivalence takes place because $1 \in Y$.

b) If Y is a maximal element in $\mathcal{P}'$ and $R \neq \{0\}$, then the ring $(Y, \oplus, \cdot)$ is a subring of R if and only if the characteristic of R is 2.

The condition (4) is verified if and only if $2x = 0$ for $x = 1 \in R$ which implies that R has the characteristic 2.

c) Let K be a field of characteristic greater than 2 (in particular, K could be $\mathbb{R}$ or $\mathbb{C}$) and $R = EndV$. In this case, if Y is a maximal element in $\mathcal{P}'$, then the ring $(Y, \oplus, \cdot)$ is not a subring of R.

We know that if $\alpha \in K$ then the function

$$t_\alpha : V \to V,\ t_\alpha(x) = \alpha x$$

is an endomorphism of V, i.e. $t_\alpha \in EndV = R$, and $\varphi : K \to R$, $\varphi(\alpha) = t_\alpha$ is a unitary and injective homomorphism of rings.

So the characteristic of R concides to the characteristic of K, so R has a characteristic different from 2.

d) **Stone's Theorem.** If $(R, +, \cdot)$ is a Boole ring then R is a Boole lattice with respect to the operations " $\vee$ " and " $\wedge$ " defined by

$$x \vee y = x + y - xy \text{ and } x \wedge y = xy$$

The correspondence $(R, +, \cdot) \longmapsto (R, \vee, \wedge)$ introduces a bijection between the class of Boole rings and the class of Boole lattices.

On the other side, if $(R, \vee, \wedge)$ is a Boole lattice then R is a Boole ring with respect to the operations " $+$ " and " $\cdot$ " defined by

$$x + y = (x \wedge y') \vee (x' \wedge y) \text{ and } xy = x \wedge y$$

If we denote the above bijection by φ, we have $\varphi^{-1}(R, \vee, \wedge) = (R, +, \cdot)$

This theorem results from Theorems 2 and 3 and from Corollary a), taking into account that in this case $\mathcal{P}' = \mathcal{P}(R)$ and $Y = R$.

Theorem 4. If R is a ring with unit and $\phi \neq X \in \mathcal{P}'$, i.e. X is a non empty subset of R composed by idempotents which commute, then the elements of R such as

$$x_{11}x_{12}\dots x_{1n_1} \vee x_{21}x_{22}\dots x_{2n_2} \vee \dots \vee x_{k1}x_{k2}\dots x_{kn_k} \tag{6}$$

where $x_{ij} \in X$ $(i = 1, \dots, k; j = 1, \dots, n_k; k, n_k \in \mathbb{N}^*)$ are idempotents and commute between them.

The set $\overline{X}$ of elements such as (6) is a distributive lattice with respect to the relation " $\leq$ ". This lattice is generated by X.

Proof. From Theorem 1 it results that there exists a maximal element $Y \in \mathcal{P}'$ such that $X \subseteq Y$ and from Theorem 2 we know that $(Y, \leq)$ is a distributive lattice.

From $X \subseteq Y$ and the fact that $(Y, \leq)$ is a lattice, it results that $\overline{X} \subseteq Y$.

So elements such as (6) are idempotents and commute between them. If y, z are elements such as (6), that is $y, z \in \overline{X}$, then it is obvious that $y \vee z \in \overline{X}$ and from the distributivity of the operation " $\wedge$ " (which coincides with " $\cdot$ ") with respect to " $\vee$ ", it results that yz is also such as (6).

This means that $\overline{X}$ is a sublattice of $(Y, \leq)$.

From (6) it results that $X \subseteq \overline{X}$. If Z is a sublattice of Y and $X \subseteq Z$, then from (6) it follows that $\overline{X} \subseteq Z$.

Thus we have proved that $\overline{X}$ is the least sublattice of Y which includes X.

So $(\overline{X}, \leq)$ is a distributive lattice generated by X.

Remarks. Considering $R = EndV$ in Theorem 4, where V is a linear space over $\mathbb{R}$ or $\mathbb{C}$, we obtain Propositions 6 and 7, section 1.2 from [1].

Theorem 5. If R is a ring with unit and $\phi \neq X \in \mathcal{P}'$ and $X^c = \{1 - x | x \in X\}$ then the elements such as (6) where $x_{ij} \in X \cup X^c$, $(i = 1, \dots, k;\ j = 1, \dots, n_k;\ k, n_k \in \mathbb{N}^*)$ are idempotents which commute and the set $\overline{\overline{X}}$ of this elements is a Boole lattice generated by X.

Proof. If $x, y \in X \cup X^c$ then we can easily prove that x and y are idempotents and $xy = yx$. Using Theorem 1 it follows that there exists a maximal element $Y \in \mathcal{P}'$ such that $X \cup X^c \subseteq Y$.

From Theorem 2 it results that $(Y, \leq)$ is a Boole lattice and from the proof of Theorem 4 it follows that $\overline{\overline{X}}$ is the sublattice of Y generated by $X \cup X^c$.

So $\overline{\overline{X}}$ is a distributive lattice. If $x \in X$ then

$$\begin{aligned} x \wedge (1 - x) &= x - x^2 = x - x = 0, \\ x \vee (1 - x) &= x + 1 - x - x(1 - x) = 1 - x + x^2 = 1 - x + x = 1 \end{aligned}$$

which shows that $0, 1 \in \overline{\overline{X}}$.

Using de Morgan's formulas in the Boole lattice Y

$$(y \vee z)' = y' \wedge z' \text{ and } (y \wedge z)' = y' \vee z'$$

for $\forall y, z \in Y$, and using the distributiveness, it follows that the complement of an element of $\overline{\overline{X}}$ is also in $\overline{\overline{X}}$.

So $\overline{\overline{X}}$ is a Boole sublattice of the lattice Y.

We have $X \subseteq \overline{\overline{X}}$ and if Z is a Boole sublattice of Y which includes X then $\overline{\overline{X}} \subseteq Z$.

This means that $\overline{\overline{X}}$ is the smallest Boole lattice including X that is $\overline{\overline{X}}$ is generated by X.

Remarks. Considering, in Theorem 5, $R = EndV$ where V is a linear space over $\mathbb{R}$ or $\mathbb{C}$ it results the construction of the Boole algebra of projectors, given in section 1.4. of [1].

References

[1] F. J. Delvos, W. Schempp, *Boolean methods in interpolation and approximation*, John Wiley& Sons, New York, 1989.

[2] F. J. Delvos, *d-variate Boolean interpolation*, Journal of Approximation Theory, **34**(1982) 99-44.

[3] W. J. Gordon, *Distributive lattices and approximation of multivariate functions*, In "Proc. Symp. Approximation with Special Emphasis on Spline Functions" (Madison, Wisc. 1969) (I. J. Schoenberg, ed.), 223-277.

[4] W. J. Gordon, E. W. Cheney, *Bivariate and multivariate interpolation with noncommutative projectors*, In "Linear Spaces and Approximation" (P. L. Butzer and B. Sz. Nagy, eds.), ISNM 40, Birkhauser, Basel, 1977, 381-387.

[5] M. H. Stone, *The theory of representation for Boolean algebras*, Trans. Amer. Math. Soc., **40**(1936), 37-111.

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