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MINIMAL CURVES IN ALMOST MINKOWSKI MANIFOLD

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Abstract. In a Lorentz manifold $[M, g]$ with a global timelike vector field Z which respects $g(Z, Z) = -1$ and its distribution is involutive, we consider a topological norm and this corresponding length of curves. We find the local equations of minimal curve of this length functional.

1. Introduction

Let M be a $(n+1)$ dimensional connected paracompact without boundary manifold and g a nondegenerate bilinear form with diagonal form +,+,...,+,- to each tangent space.

Given a global vector field Z so that $g(Z, Z) = -1$ on M, we say that the structure (M, g, Z) is a *time-normalized space-time manifold.*

Definition 1.1. A time-normalized space-time manifold (M, g, Z) is an *almost Minkowski manifold* if the distribution:

$$x \in M \longmapsto \{X \in T_x M \mid g(X, Z) = 0\} \text{ is involutive.}$$

Definition 1.2. A time-normalized space-time manifold (M, g, Z) for which there is $f : M \to \mathbf{R}$ so that $Z = \nabla f$ is called a *functional normalized space-time manifold* and it is noted (M, g, f).

Remark 1. Obviously any functional normalized space-time manifold is an almost Minkowski manifold.

In [5] the necessary and sufficient conditions for a time-normalized space-time manifold are given to be an almost Minkowski manifold.

Proposition 1.1. Let (M, g, Z) be a time-normalized space-time manifold with $H^1(M) = \{0\}$. The necessary and sufficient conditions for the existence of an atlas $\overline{A}$ of M, so that the local coordinates of g respects:

$$\partial_{n+1} = Z \text{ and } \frac{\partial g_{an+1}}{\partial x^b} = \frac{\partial g_{bn+1}}{\partial x^a},\ \forall a, b = \overline{1, n+1} \quad (1)$$

is (M, g, Z) to be a functional normalized space-time manifold.

Proof. We define the $1-$form $\omega = g_{an+1} dx^a$. Then:

$$d\omega = \left(\frac{\partial g_{an+1}}{\partial x^b} - \frac{\partial g_{bn+1}}{\partial x^a} \right) dx^a \wedge dx^b$$

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and the condition (1) implies $d\omega = 0$ and by hypothesis ω is an exact 1-form and there exist $f : M \to \mathbf{R}$ so that:

$$df = \omega \Leftrightarrow \frac{\partial f}{\partial x^a} dx^a = g_{an+1} dx^a \text{ or}$$

$$g^{ab} \frac{\partial f}{\partial x^a} = \delta^b_{n+1} \Rightarrow g^{ab} \frac{\partial f}{\partial x^a} \partial_b = \partial_{n+1} \Rightarrow \nabla f = Z$$

Reciprocally if $\nabla f = Z$ then $g_{an+1} = \dfrac{\partial f}{\partial x^a}$ and $\dfrac{\partial g_{an+1}}{\partial x^b} = \dfrac{\partial g_{bn+1}}{\partial x^a}$

Remark 2. The restrictive condition of almost Minkowski manifold does not imply the chronologicity. For example if $M = S^1 \times \mathbf{R}$ with $g = -d\theta^2 + dt^2$ admits the curve $\gamma(s) = (s, t_0)$ which is a closed timelike curve, and $\left(M, g; \frac{\partial}{\partial \theta}\right)$ is a almost Minkowski manifold.

If $Lor(M)$ denotes the set of Lorentz metrics. with partial ordering relation:

$$g_{1 \leq} g_2 \Leftrightarrow \forall p \in M,\ \forall X \in T_p M,\ g_1(X, X) \leq 0 \Rightarrow g_2(X, X) \leq 0$$

then by [1, Prop. 6.4.9] the functional-normalized space-time manifold can be characterize by the following statement:

Proposition 1.2. A Lorentz manifold (M, g) can be become a functional normalization Lorentz manifold if and only if a causal metric g_1 exists, so that $g \leq g_1$.

2. Minimal curve in almost Minkowski manifold

Definition 2.1. We call $Z-$*norm* on a almost Minkowski manifold the application: $|\ |_Z : TM \to \mathbf{R}$, defined by:

$$|X|_Z = |g(X, Z)| + \sqrt{g(X, Z)^2 + g(X, X)}$$

Remark 1. a) It is proved in [5] that:

$$|X|_Z = \min\{\lambda \geq 0,\ -\lambda Z_x \leq X \leq \lambda Z_x\}$$

where the ordering relation on $T_x M$ is defined by:

$$X \leq Y \Leftrightarrow (X = Y) \vee (g(Y - X, Y - X) < 0 \wedge g(Z_x, Y - X) < 0)$$

b) For a almost Minkowski manifold (M, g, Z), the expression of the norm in the preferential atlas $\overline{A}$ (which exists [5]) with:

$$\begin{cases} \partial_{n+1} = Z \\ g_{an+1} = -\delta^a_{n+1}, \forall a = \overline{1, n+1} \end{cases}$$

is:

$$|X|_Z = \left|X^{n+1}\right| + \sqrt{g_{ij} X^i X^j}, \forall i, j = \overline{1, n} \text{ where } X = X^a \partial_a, a = \overline{1, n+1}$$

If $p_1,\ p_2 \in M$, we note the $\Omega_{p_1 p_2}$ the set of $\mathcal{C}^\infty$ curves from p_1 to p_2 and its subsets:

$$\Omega^+_{p_1 p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) > 0,\ \forall t \in [\alpha, \beta]\}$$
$$\Omega^-_{p_1 p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) < 0,\ \forall t \in [\alpha, \beta]\}$$
$$\Omega^0_{p_1 p_2} = \{\gamma : [\alpha, \beta] \to M,\ g\left(\gamma'(t), Z_{\gamma(t)}\right) = 0,\ \forall t \in [\alpha, \beta]\}$$

Definition 2.2. For $\gamma \in \Omega_{p_1 p_2}$ we define the $Z-$length of γ by:

$$L_Z(\gamma) = \int_\alpha^\beta |\gamma'(t)|_Z\, dt$$

Theorem 2.1. If $\gamma_0 \in \Omega^+_{p_1p_2}$ exist so that:

$$L_Z(\gamma_0) \leq L_Z(\gamma), \ \forall \gamma \in \Omega^+_{p_1p_2}$$

then there is a parametrization of γ_0 which in local preferential coordinates verifies:

$$h_{ab}\frac{d^2x_0^b}{ds^2} + \frac{1}{2}\left[\frac{\partial h_{ab}}{\partial x^c} + \frac{\partial h_{ac}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^a}\right]\frac{dx_0^b}{ds}\frac{dx_0^c}{ds} = 0, \ \forall a = \overline{1, n+1}$$

where:

$$h_{ab} = \begin{cases} g_{ab} \text{ if } (a,b) \neq (n+1, n+1) \\ 0 \text{ if } (a,b) = (n+1, n+1) \end{cases}$$

and the local equations of γ_0 are:

$$\begin{cases} x^a = x_0^a(s) \\ a = \overline{1, n+1}, \ s \in [\alpha', \beta'] \end{cases} \quad \text{with} \quad \left(g\left(\frac{d\gamma}{ds}, Z\right)^2 + g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = 1\right)$$

Proof. Let be $\gamma \in \Omega^+_{p_1p_2}$ with the local preferential atlas $\overline{A}: \begin{cases} x^a = x^a(t) \\ a = \overline{1, n+1} \end{cases}$, $t \in [\alpha, \beta]$ Then $g\left(\frac{d\gamma}{dt}, Z_{\gamma(t)}\right) > 0$ implies $\frac{dx^{n+1}}{dt} < 0$. and the Z-length functional is:

$$L_Z(\gamma) = \int_\alpha^\beta \left\{-\frac{dx^{n+1}}{dt} + \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\right\} dt \text{ where } i, j \in \overline{1, n}$$

We denote:

$$G\left(t, (x^a), \left(x^{a'}\right)\right) = -\frac{dx^{n+1}}{dt} + \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}$$

and we calculate:

$$\frac{\partial G}{\partial x^m} = \frac{\frac{\partial g_{ij}}{\partial x^m}\frac{dx^i}{dt}\frac{dx^j}{dt}}{2\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}} \ ; \ \frac{\partial G}{\partial x^{m'}} = +\delta_m^{n+1} + \frac{g_{im}\frac{dx^i}{dt}\left(1 - \delta_m^{n+1}\right)}{\sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}}$$

Because $(g_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ is a matrix of a positive definite bilinear form, it is possible to find a parametrization of γ so that $g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1$ or $g\left(\frac{d\gamma}{ds}, Z\right)^2 + g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = 1$. Then the Euler-Lagrange system of the Z-length functional for the curves of $\Omega^+_{p_1p_2}$ becomes:

$$\frac{\partial G}{\partial x^m} - \frac{d}{ds}\left(\frac{\partial G}{\partial x^{m'}}\right) = 0$$

or

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^m}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{d}{ds}\left[-\delta_m^{n+1} + g_{im}\frac{dx^i}{ds}\left(1 - \delta_m^{n+1}\right)\right] = 0.$$

For $m \neq n+1$ we have:

$$g_{im}\frac{d^2x^i}{ds^2} + \frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{\partial g_{im}}{\partial x^{n+1}}\frac{dx^i}{ds}\frac{dx^{n+1}}{ds} = 0$$

or

$$g_{im}\frac{d^2x^i}{ds^2} + \frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} +$$
$$+\frac{1}{2}\left[\frac{\partial g_{im}}{\partial x^{n+1}} + \frac{\partial g_{n+1m}}{\partial x^i} - \frac{\partial g_{in+1}}{\partial x^m}\right]\frac{dx^i}{ds}\frac{dx^j}{ds} +$$

$$+\frac{1}{2}\left[\frac{\partial g_{jm}}{\partial x^{n+1}}+\frac{\partial g_{n+1m}}{\partial x^{j}}-\frac{\partial g_{n+1j}}{\partial x^{m}}\right]\frac{dx^{m+1}}{ds}\frac{dx^{j}}{ds}+$$

$$+\frac{1}{2}\left[\frac{\partial g_{n+1m}}{\partial x^{n+1}}+\frac{\partial g_{n+1m}}{\partial x^{n+1}}-\frac{\partial g_{n+1n+1}}{\partial x^{m}}\right]\frac{dx^{n+1}}{ds}\frac{dx^{n+1}}{ds}=0$$

that is:

$$g_{am}\frac{d^2x^a}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{bm}}{\partial x^{a}}+\frac{\partial g_{am}}{\partial x^{b}}-\frac{\partial g_{ab}}{\partial x^{m}}\right]\frac{dx^{a}}{ds}\frac{dx^{b}}{ds}=0 \tag{2}$$

where $a,b\in\overline{1,n+1}$

For $m=n+1$ the Euler-Lagrange equation become:

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^{m}}\frac{dx^{i}}{ds}\frac{dx^{j}}{ds}=0$$

or

$$\left(g_{n+1a}-g_{n+1n+1}\right)\frac{d^2x^a}{ds^2}+\frac{1}{2}\left[\frac{\partial g_{bn+1}}{\partial x^{a}}+\frac{\partial g_{an+1}}{\partial x^{b}}-\frac{\partial g_{ab}}{\partial x^{n+1}}\right]\frac{dx^{a}}{ds}\frac{dx^{b}}{ds}=0 \tag{3}$$

Therefore the relations 2 and 3 implies:

$$h_{ab}\frac{d^2x_0^b}{ds^2}+\frac{1}{2}\left[\frac{\partial h_{ab}}{\partial x^{c}}+\frac{\partial h_{ac}}{\partial x^{b}}-\frac{\partial h_{bc}}{\partial x^{a}}\right]\frac{dx_0^{b}}{ds}\frac{dx_0^{c}}{ds}=0,\ \forall a=\overline{1,n+1}$$

where:

$$h_{ab}=\begin{cases} g_{ab} \text{ if } (a,b)\neq(n+1,n+1)\\ 0 \text{ if } (a,b)=(n+1,n+1)\end{cases}$$

Remark 2. The analogous statement for the case $\Omega^{-}_{p_1p_2}\neq\phi$. For $\Omega^{0}_{p_1p_2}\neq\phi$ the equation $g\left(\frac{d\gamma}{dt},Z\right)=0$ becomes $-x^{n+1'}=0$ hence
$x^{n+1}=\mathbf{k}$, where $\mathbf{k}$ is a constant. If exist $\gamma_0\in\Omega^{0}_{p_1p_2}$ so that

$$L_Z(\gamma_0)\leq L_Z(\gamma),\ \forall\gamma\in\Omega^{0}_{p_1p_2}$$

than we can find a parametrization of γ_0 so that its local preferential coordinates verify:

$$\begin{cases} \widetilde{h}_{ik}\frac{d^2x^i}{ds^2}+\frac{1}{2}\left[\frac{\partial\widetilde{h}_{ik}}{\partial x^{j}}+\frac{\partial\widetilde{h}_{jk}}{\partial x^{i}}-\frac{\partial\widetilde{h}_{ij}}{\partial x^{k}}\right]\frac{dx^i}{ds}\frac{dx^j}{ds}=0\\ \frac{dx^{n+1}}{ds}=0,\ \forall i,j,k=\overline{1,n}\end{cases}$$

where $\widetilde{h}_{ij}=g_{ij}\left(x^1,x^2,..,x^n,\mathbf{k}\right)$.

If p_1,p_2 are on the same pure timelike curve, meaning that:

$\exists\gamma_0:[\alpha,\beta]\to M,\ \gamma(\alpha)=p_1,\ \gamma(\beta)=p_2,\ \gamma'(t)=\lambda(t)Z_{\gamma(t)}$, where $\lambda(t)>0$ or $\lambda(t)<0$, we can find a parametrization of γ_0 in the preferential coordinates, so that:

$$\frac{dx_0^a}{dt}=\pm\delta^a_{n+1},\ \forall a=\overline{1,n+1} \text{ and:}$$

$$\begin{aligned} L_Z(\gamma_0) &= \left|x_0^{n+1}(\beta)-x_0^{n+1}(\alpha)\right|\leq\\ &\leq \int_\alpha^\beta\left\{\left|\frac{dx^{n+1}(t)}{dt}\right|+\sqrt{g_{ij}\frac{dx^i(t)}{dt}\frac{dx^j(t)}{dt}}\right\}dt=L_Z(\gamma)\end{aligned}$$

for every $\gamma\in\Omega^{+}_{p_1p_2}$ if $\lambda>0$ and $\gamma\in\Omega^{-}_{p_1p_2}$ if $\lambda<0$.

Corollary 2.1. Let (M,g,f) be a almost Minkowski manifold. For any $p_1\in M$,there is a neighborhood V_1, so that for any $p_2\in V_1$ with $\Omega^{+}_{p_1p_2}\neq\Phi$ there is at least a curve $\gamma_0\in\Omega^{+}_{p_1p_2}$ which is minimal for the $Z-$length functional

Remark 3. We can state the same results for $\Omega^{-}_{p_1 p_2}$ and $\Omega^{0}_{p_1 p_2}$.

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