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ON A GENERAL CLASS OF GAMMA APPROXIMATING OPERATORS

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Abstract. Many authors introduced and studied positive linear operators, using Euler's gamma function Γ_p, $p > 0$. We shall define a more general linear transform $\Gamma_p^{(a,b)}$, $a, b \in \mathbb{R}$, from which we obtain as particular cases the gamma first-kind transform and the gamma second-kind transform. For different values of a and b we obtain several gamma type operators studied in the literature.

1. **Introduction**

Many authors introduced and studied positive linear operators, using Euler's gamma function: [3], [4], [7], [8], [9].

We shall define a more general linear transform from which we obtain as particular cases the gamma first-kind transform and the gamma second-kind transform.

Euler's gamma function is defined for $p > 0$ by the following formula

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt \tag{1.1}$$

which can be written as

$$\Gamma(p) = \int_0^1 \ln^{p-1}\left(\frac{1}{u}\right) du \tag{1.2}$$

For $a, b \in \mathbb{R}$ we define the (a, b)-gamma transform of a function f by the functional (see also [5])

$$(\Gamma_p^{(a,b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f(xe^{-bt}(t/p)^a) dt \tag{1.3}$$

where Γ is defined by (1.1) (or (1.2)) and $f \in L_{1,loc}(0, \infty)$ such that $\Gamma_p^{(a,b)}|f| < \infty$. The above relation is equivalent with

$$(\Gamma_p^{(a,b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^1 \ln^{p-1}\left(\frac{1}{u}\right) f\left(xu^b\left(\frac{1}{p}\ln\frac{1}{u}\right)^a\right) du \tag{1.4}$$

For different values of a and b we obtain several gamma type operators studied by many authors.

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2. The Gamma first-kind transform

If we put in (1.3) $b = 0$ we obtain the gamma first-kind transform of function f

$$(\Gamma_p^{(a)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(x\left(\frac{t}{p}\right)^a\right) dt \tag{2.1}$$

where $f \in L_{1,loc}[0,\infty)$ such that $\Gamma_p^{(a)}|f| < \infty$.

One observes that $\Gamma_p^{(a)}$ is a positive linear functional.

We state and prove:

Lemma 2.1. *The moment of order k of the functional $\Gamma_p^{(a)}$ has the following value*

$$(\Gamma_p^{(a)} e_k)(x) = \frac{\Gamma(p+ka)}{p^{ka}\Gamma(p)} x^k, \quad x > 0. \tag{2.2}$$

Proof. By using (1.1) we easily obtain

$$(\Gamma_p^{(a)} e_k)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} x^k \left(\frac{t}{p}\right)^{ka} dt$$

$$= \frac{x^k}{\Gamma(p)p^{ka}} \int_0^\infty e^{-t} t^{p+ka-1} dt = \frac{\Gamma(p+ka)}{p^{ka}\Gamma(p)} x^k. \ \square$$

Consequently we obtain

$$\Gamma_p^{(a)} e_1(x) = \frac{\Gamma(a+p)}{p^a \Gamma(p)} x, \quad \Gamma_p^{(a)} e_2(x) = \frac{\Gamma(p+2a)}{p^{2a}\Gamma(p)} x^2 \tag{2.3}$$

Particular cases

Case 1. If we consider $a = 1$ in (2.1) we obtain

$$(\Gamma_p f)(x) = (\Gamma_p^{(1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f\left(\frac{xt}{p}\right) dt \tag{2.4}$$

For $a = 1$, Lemma 2.1 leads us to the following

Corollary 2.2. *The moment of order k, $k \in \mathbb{N}$, of the functional Γ_p has the following values*

$$(\Gamma_p e_k)(x) = \frac{\Gamma(p+k)}{\Gamma(p)p^k} x^k, \quad x > 0.$$

We deduce

$$(\Gamma_p e_1)(x) = x, \quad (\Gamma_p e_2)(x) = \frac{p+1}{p} x^2, \quad \Gamma_p((t-x)^2; x) = \frac{x^2}{p}.$$

If we choose $p = n$, $n \in \mathbb{N}$ in (2.4) then we obtain Post-Wider's positive linear operator, defined for $f \in L_{1,loc}(0,\infty)$ by

$$(P_n f)(x) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-t} t^{n-1} f\left(\frac{xt}{n}\right) dt. \tag{2.5}$$

If we replace p by nx, for $n \in \mathbb{N}$ and $x \geq 0$, in (2.4), we reobtain Rathore's positive linear operator [8], defined for $f \in L_{1,loc}(0,\infty)$ by

$$(R_n f)(x) = \frac{1}{\Gamma(nx)} \int_0^\infty e^{-t} t^{nx-1} f\left(\frac{t}{n}\right) dt. \tag{2.6}$$

Corollary 2.3. *One has*

$$P_n((t-x)^2;x) = \frac{x^2}{n}, \quad R_n((t-x)^2;x) = \frac{x}{n}.$$

Proof. It is obtained from Corollary 2.2.

Szasz's operator is defined by the following formula

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{2.7}$$

If we apply gamma transform (2.4) to Szasz's operator we obtain the following positive linear operator.

Theorem 2.4. *The following identity*

$$\Gamma_p(S_n f)(x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{p^p (p)_k}{(nx+p)^{p+k}} f\left(\frac{k}{n}\right), \quad x > 0 \tag{2.8}$$

holds true. Here $(p)_0 = 1$ *and* $(p)_k = p(p+1)\dots(p+k-1)$, $k \geq 1$.

Proof.

$$\begin{aligned}
\Gamma_p(S_n f)(x) &= \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} e^{-\frac{ntx}{p}} \sum_{k=0}^{\infty} \left(\frac{ntx}{p}\right)^k \frac{1}{k!} f\left(\frac{k}{n}\right) \\
&= \frac{1}{\Gamma(p)} \sum_{k=0}^{\infty} \frac{(nx)^k}{p^k k!} f\left(\frac{k}{n}\right) \int_0^{\infty} e^{-t\left(\frac{nx}{p}+1\right)} t^{p+k-1} dt \\
&= \frac{1}{\Gamma(p)} \sum_{k=0}^{\infty} \frac{(nx)^k}{p^k k!} f\left(\frac{k}{n}\right) \Gamma(p+k) \left(\frac{p}{nx+p}\right)^{p+k} \\
&= \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{p^p (p)_k}{(nx+p)^{p+k}} f\left(\frac{k}{n}\right).
\end{aligned}$$

In [2] A. Lupaş considered the operator L_n, defined for $f \in C[0,\infty)$ by

$$(L_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) \tag{2.9}$$

which reproduces linear functions.

This operator is similar with Szasz's operator. In [2] the author asks to find properties of operator L_n. Some approximation properties were given in [1]. In the following theorem we shall prove that this operator can be obtained by the composite of Rathore's operator with Szasz's operator.

Theorem 2.5. *a) If* P_n *is the Post-Wider's operator (2.5) then* $B_n^* f = P_n(S_n f)$, *where* B_n^* *is the Baskakov's operator*

$$(B_n^* f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

b) If R_n *is the Rathore's operator (2.6) then*

$$L_n f = R_n(S_n f).$$

Proof. The proof is obtained from Theorem 2.4 for $p = n$ in the first case and for $p = nx$ in the second case.

Corollary 2.6. *The operator L_n can be written in the following manner*

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(nx)_k}{n^k} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right].$$

Proof. We apply Theorem 2.4(b), using for the Szasz's operator the following formula

$$(S_n f)(x) = \sum_{k=0}^{\infty} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] x^k.$$

Case 2. If we replace $a = -1$ in (2.1) we obtain the following gamma transform

$$(\widetilde{\Gamma}_p f)(x) = (\Gamma_p^{(-1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} f\left(\frac{px}{t}\right) dt \tag{2.10}$$

where Γ is the gamma function (1.1), $p > 0$, and $f \in L_{1,loc}(0,\infty)$ such that $\widetilde{\Gamma}_p |f| < \infty$.

One observes that $\widetilde{\Gamma}_p$ is a positive linear functional.

Lemma 2.7. *The moment of order k, $k \in \mathbb{N}$, $k < p$, of the functional $\widetilde{\Gamma}_p$ has the following value*

$$\widetilde{\Gamma}_p e_k(x) = \frac{\Gamma(p-k)}{\Gamma(p)} (px)^k, \quad x > 0.$$

Proof. It is obtained from Lemma 2.1, for $a = -1$.

We deduce

$$(\widetilde{\Gamma}_p e_2)(x) = x^2 + \frac{x^2}{p-1}; \quad \widetilde{\Gamma}_p((t-x)^2; x) = \frac{x^2}{p-1}.$$

If we put $p = n + 1$ in (2.9) we obtain the gamma operator introduced and studied by A. Lupaş and M. Müller [4]

$$(G_n f)(x) = \frac{1}{n!} \int_0^{\infty} e^{-t} t^n f\left(\frac{(n+1)x}{t}\right) dt \tag{2.11}$$

Corollary 2.8.

$$G_n((t-x)^2; x) = \frac{x^2}{n}.$$

Proof. It is obtained from Lemma 2.7 for $p = n + 1$.

Several papers have dealt with these operators: [3], [4], [9].

3. **The Gamma second-kind transform**

If we choose in (1.3) $a = 0$ then we obtain the gamma second-kind transform of a function f

$$(\Gamma_p^{(b)} f)(x) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-t} t^{p-1} f(x e^{-bt}) dt \tag{3.1}$$

where Γ is the gamma function (1.1), $p > 0$, and $f \in L_{1,loc}[0,\infty)$ such that $\Gamma_p^{(b)} |f| < \infty$.

We consider here only the case $b = 1$.

$$(\Gamma_p^* f)(x) = (\Gamma_p^{(1)} f)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} f(xe^{-t}) dt \tag{3.2}$$

Formula (3.2) is equivalent with (see (1.4))

$$(\Gamma_p^* f)(x) = \frac{1}{\Gamma(p)} \int_0^1 \ln^{p-1} \frac{1}{u} f(ux) du$$

Clearly, Γ_p^* is a positive linear functional.

Lemma 3.1. *The moment of order k, $k \in \mathbb{N}$, of the functional Γ_p^* has the following value*

$$(\Gamma_p^* e_k)(x) = \frac{x^k}{(k+1)^p}$$

Proof. We can write successively

$$(\Gamma_p^* e_k)(x) = \frac{1}{\Gamma(p)} \int_0^\infty e^{-t} t^{p-1} (xe^{-t})^k dt = \frac{x^k}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-t(k+1)} dt =$$

$$= \frac{x^k}{\Gamma(p)} \frac{\Gamma(p)}{(k+1)^p} = \frac{x^k}{(k+1)^p}.$$

By using (3.1), for $p = \alpha$, $\alpha > 0$ we obtain the positive linear operator

$$(A_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} f(xe^{-t}) dt, \tag{3.3}$$

or equivalent (see (3.2))

$$(A_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \ln^{\alpha-1} \frac{1}{t} f(tx) dt. \tag{3.4}$$

This operator was introduced by the author in [5] and it is strongly related with Cesaro means of order α (see [5]). This operator is an approximating operator for $\alpha \to 0$, for example, $\alpha = 1/n$, $n \in \mathbb{N}$.

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