CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper we define and study new classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ of univalent functions with negative coefficients.

1. Introduction

Let $\mathbf{U}$ denote the open unit disc: $\mathbf{U} = \{z\ ;\ z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathbf{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic and univalent in $\mathbf{U}$.

For $f \in \mathbf{S}$ we define the differential operator $\mathbf{D}^n$ (Sălăgean [1])

$$\begin{aligned} \mathbf{D}^0 f(z) &= f(z) \\ \mathbf{D}^1 f(z) &= \mathbf{D}f(z) = zf'(z) \end{aligned}$$

and

$$\mathbf{D}^n f(z) = \mathbf{D}(\mathbf{D}^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1, 2, 3, ...\}.$$

We note that if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$\mathbf{D}^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \ ;\ z \in \mathbf{U}.$$

Let T denote the subclass of $\mathbf{S}$ which can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k \tag{1}$$

We say that a function $f \in T$ is in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ $\quad 0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$ if

$$\left| \frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma \left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha\right] - B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1\right]} \right| < \beta \quad , \quad z \in \mathbf{U} \tag{2}$$

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$$\frac{B}{B-A} < \gamma \leq \begin{cases} \dfrac{B}{(B-A)\alpha} & ; \quad \alpha \neq 0 \\ 1 & ; \quad \alpha = 0 \end{cases} \tag{3}$$

where

$$F_{n,\lambda}(z) = (1-\lambda)D^n f(z) + \lambda D^{n+1} f(z) \ \ ; \ \lambda \geq 0 \ \ ; \ f \in T \tag{4}$$

Remark 1.

$$\begin{aligned} F_{0,\lambda}(z) &= z - \sum_{k=2}^{\infty} [1+(k-1)\lambda] \, |a_k| \, z^k \\ F_{1,\lambda}(z) &= z - \sum_{k=2}^{\infty} k \, [1+(k-1)\lambda] \, |a_k| \, z^k \\ &\ldots\ldots\ldots\ldots\ldots \\ F_{n,\lambda}(z) &= z - \sum_{k=2}^{\infty} k^n \, [1+(k-1)\lambda] \, |a_k| \, z^k \end{aligned} \tag{5}$$

For $n = 0$, $T_{0,\lambda}(A,B,\alpha,\beta,\gamma) = T^*_\lambda(A,B,\alpha,\beta,\gamma)$ *and for* $n = 1$, $T_{1,\lambda}(A,B,\alpha,\beta,\gamma) = C^*_\lambda(A,B,\alpha,\beta,\gamma)$.

The class $T^*_\lambda(A,B,\alpha,\beta,\gamma)$ *and* $C^*_\lambda(A,B,\alpha,\beta,\gamma)$ *was studied by S.B.Joshi and H.M.Srivastava* [3] *and S.B.Joshi* [2].

2. Characterization theorem

Theorem 2. *Let* $f \in T$, $f(z) = z - \sum\limits_{k=2}^{\infty} |a_k| \, z^k$. *Then* $f(z)$ *is in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty} |a_k| \, k^n \, [1+\lambda(k-1)] \, \{(k-1) + \beta \, [(B-A)\gamma(k-\alpha) - B(k-1)]\} \leq \\ \leq \ & \beta\gamma(B-A)(1-\alpha) \end{aligned} \tag{6}$$

and the result is sharp.

If we denote

$$\begin{aligned} &D_n \, (k,A,B,\alpha,\beta,\gamma,\lambda) = \\ = \ & k^n \, [1+\lambda \, (k-1)] \, \{(k-1) + \beta \, [(B-A) \, \gamma \, (k-\alpha) - B \, (k-1)]\} \end{aligned} \tag{7}$$

then (6) becomes

$$\sum_{k=2}^{\infty} |a_k| \, D_n \, (k,A,B,\alpha,\beta,\gamma,\lambda) \leq \beta\gamma(B-A)(1-\alpha) \tag{8}$$

Proof. Assume that

$$\begin{aligned} &\sum_{k=2}^{\infty} |a_k| \, k^n \, [1+\lambda(k-1)] \, \{(k-1) + \beta \, [(B-A)\gamma(k-\alpha) - B(k-1)]\} \leq \\ \leq \ & \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

and let $|z| = 1$. Then we have

$$\begin{aligned}
& \left|zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right| - \\
& -\beta\left|(B-A)\gamma\left[zF'_{n,\lambda}(z) - \alpha F_{n,\lambda}(z)\right] - B\left[zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right]\right| \\
= & \left|zF'_{n,\lambda}(z) - F_{n,\lambda}(z)\right| - \\
& -\beta\left|\left[(B-A)\gamma - B\right] zF'_{n,\lambda}(z) + \left[B - (B-A)\gamma\alpha\right] F_{n,\lambda}(z)\right| \\
= & \left|\sum_{k=2}^{\infty} k^n (1-k)\left[1+(k-1)\lambda\right]|a_k| z^k\right| - \\
& -\beta\left|\left[(B-A)\gamma - B\right] z - \left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k| z^k + \right. \\
& \left. + \left[B-(B-A)\gamma\alpha\right] z - \left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k| z^k\right| \\
= & \left|\sum_{k=2}^{\infty} k^n (1-k)\left[1+(k-1)\lambda\right]|a_k| z^k\right| - \\
& -\beta\left|(B-A)\gamma(1-\alpha) z - \left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k| z^k - \right. \\
& \left. - \left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k| z^k\right| \\
\leq & \sum_{k=2}^{\infty} k^n (k-1)\left[1+(k-1)\lambda\right]|a_k| |z|^k - \beta(B-A)\gamma(1-\alpha)|z| + \\
& +\beta\left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k| |z|^k \\
& +\beta\left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k| |z|^k \\
\leq & \sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k|\left\{(k-1) + \beta\left[(B-A)\gamma(k-\alpha) - B(k-1)\right]\right\} - \\
& -\beta\gamma(B-A)(1-\alpha) \leq 0
\end{aligned}$$

Consequently, by the maximum modulus theorem , the functions $f(z)$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Conversely, assume that

$$\left|\frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha\right] - B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1\right]}\right| < \beta \Longleftrightarrow$$

$$\left|\sum_{k=2}^{\infty} k^n (1-k)\left[1+(k-1)\lambda\right]|a_k| z^k\right|$$

$$\leq \beta\left|(B-A)\gamma(1-\alpha)z - \left[(B-A)\gamma - B\right]\sum_{k=2}^{\infty} k^{n+1}\left[1+(k-1)\lambda\right]|a_k| z^k - \right.$$

$$\left. - \left[B-(B-A\gamma\alpha)\right]\sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k| z^k\right|$$

Since $|\mathrm{Re}\,(z)| \leq |z|$ for any z, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=2}^{\infty} k^n (k-1)\left[1+(k-1)\lambda\right]|a_k| z^k}{\beta(B-A)\gamma(1-\alpha)z - \sum\limits_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k|\left[(B-A)\gamma(k-\alpha)-B(k-1)\right]z^k}\right\} < \beta$$

Letting $z \to 1$ through real values, upon clearing the denominator in the last inequality we obtain

$$\sum_{k=2}^{\infty} k^n (k-1)\left[1+(k-1)\lambda\right]|a_k| \leq$$

$$\beta\gamma(B-A)(1-\alpha) - \sum_{k=2}^{\infty} k^n\left[1+(k-1)\lambda\right]|a_k|\,\beta\left[(B-A)\gamma(k-\alpha)-B(k-1)\right]$$

and this inequality gives the required condition.

The function

$$f(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\left\{1+\beta\left[(B-A)\gamma(2-\alpha)-B\right]\right\}}z^2$$

is an extremal function for the theorem. □

Remark 3. *For $n = 0$ and $n = 1$ the result of Theorem 1 was obtain by Joshi and Srivastava* [3].

3. **Closure Theorems**

Let the functions f_j be of the form:

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{kj}| z^k \quad ; \quad z \in \mathbf{U} \quad ; \quad j = 1, 2, ..., m \tag{9}$$

we shall prove the following results for the closure of functions in the classe $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Theorem 4. *Let the functions $f_j(z)$ defined by* (3.1) *be in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. Then the function $h(z)$, defined by*

$$h(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \quad ; \quad \textit{with } b_k = \frac{1}{m}\sum_{j=1}^{m} |a_{kj}| \tag{10}$$

also belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. As $f_j(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ it follows from Theorem 1. that

$$\sum_{k=2}^{\infty} |a_{kj}| D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha) \quad ; \quad j = 1, 2, ..., m \tag{11}$$

Therefore

$$\begin{aligned} \sum_{k=2}^{\infty} |b_k| D_n(k, A, B, \alpha, \beta, \gamma, \lambda) &= \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \frac{1}{m} \sum_{j=1}^{m} |a_{kj}| \\ &\leq \beta\gamma(B - A)(1 - \alpha) \end{aligned}$$

hence, by Theorem 1 ,

$$h(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

□

Remark 5. *For $n = 0$ we obtain Theorem 1 as Joshi*[2]. *For $n = 1$ we obtain Theorem 2 as Joshi*[2].

Theorem 6. *Let $f_j(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. Then the function $h(z)$, defined by,*

$$h(z) = \sum_{j=1}^{m} |d_j| f_j(z); \ \textit{where} \ \sum_{j=1}^{m} |d_j| = 1 \tag{12}$$

is also in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. By using definition of $h(z)$, we have

$$\begin{aligned} h(z) &= \sum_{j=1}^{m} |d_j| \left[z - \sum_{k=2}^{\infty} |a_{kj}| z^k \right] = z \sum_{j=1}^{m} |d_j| - \sum_{k=2}^{\infty} \sum_{j=1}^{m} |d_j| |a_{kj}| z^k \\ &= z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} |d_j| |a_{kj}| z^k \right) \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \left(\sum_{j=1}^{m} |d_j| |a_{kj}| \right) \\ &= \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k1}| |d_1| + \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| |d_2| + \\ &\quad ... + \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{km}| |d_m| \\ &\leq |d_1| \beta\gamma(B - A)(1 - \alpha) + |d_2| \beta\gamma(B - A)(1 - \alpha) + \\ &\quad ... + |d_m| \beta\gamma(B - A)(1 - \alpha) \\ &= \beta\gamma(B - A)(1 - \alpha) \sum_{j=1}^{m} |d_j| = \beta\gamma(B - A)(1 - \alpha) \end{aligned}$$

which implies that $h(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. □

Remark 7. *For $n = 0$ obtain Theorem 3 as Joshi*[2]. *For $n = 1$ we obtain Theorem 4 as Joshi*[2].

Theorem 8. *Let the functions*

$$f_1(z) = z - \sum_{k=2}^{\infty} |a_{k1}| z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$$

and

$$f_2(z) = z - \sum_{k=2}^{\infty} |a_{k2}| z^k \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma).$$

Then the function $p(z)$ defined by

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} (|a_{k1} + a_{k2}|) z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

Proof. Let $f_1(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ and $f_2(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$; by using Theorem 1. we get , respectively ,

$$\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k1}| \le \beta\gamma(B - A)(1 - \alpha) \tag{13}$$

and

$$\sum_{k=2}^{\infty} D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \le \beta\gamma(B - A)(1 - \alpha) \tag{14}$$

We have (see (7))

$$\begin{aligned} 2\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| &\le \sum_{k=2}^{\infty} D_{n+1}(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \le \\ &\le \beta\gamma(B - A)(1 - \alpha) \end{aligned}$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k1}| \le \frac{2}{3}\beta\gamma(B - A)(1 - \alpha)$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_{k2}| \le \frac{1}{3}\beta\gamma(B - A)(1 - \alpha) \Rightarrow$$

$$\frac{2}{3}\sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) [|a_{k1}| + |a_{k2}|] \le \beta\gamma(B - A)(1 - \alpha) \Rightarrow$$

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty} |a_{k1} + a_{k2}| z^k \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma).$$

□

4. **Integral Operators**

Theorem 9. *Let the functions $f(z)$ defined by* (1)*, be in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, and let c be a real number such that $c > -1$.*

Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{15}$$

also belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. By using the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} |b_k| z^k, where \quad |b_k| = \frac{c+1}{c+k} |a_k| \tag{16}$$

$$f \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \Rightarrow \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_k| \leq \beta\gamma(B-A)(1-\alpha)$$

$$\begin{aligned} \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |b_k| &= \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) \frac{c+1}{c+k} |a_k| < \\ &< \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_k| \\ &\leq \beta\gamma(B-A)(1-\alpha) \end{aligned}$$

$\Rightarrow F(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$. □

Theorem 10. *Let c be a real number such that $c > -1$. If $F(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ then the function $f(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

is univalent in $|z| < R$, where

$$R = \inf_k \left[\frac{D_n(k, A, B, \alpha, \beta, \gamma, \lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)(c+k)k} \right]^{\frac{1}{k-1}}, \quad k \geq 2 \tag{17}$$

The result is sharp for

$$f(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)(c+k) z^k}{D_n(k, A, B, \alpha, \beta, \gamma, \lambda)(c+1)}, \qquad k \geq 2 \tag{18}$$

Proof. Let $F(z) = z - \sum\limits_{k=2}^{\infty} |a_k| z^k$, it folows from (15) that

$$f(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} |a_k| z^k \tag{19}$$

$$F(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \Rightarrow \sum_{k=2}^{\infty} D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_k| \leq \beta\gamma(B-A)(1-\alpha) \Rightarrow$$

$$\sum_{k=2}^{\infty} \frac{D_n(k, A, B, \alpha, \beta, \gamma, \lambda) |a_k|}{\beta\gamma(B-A)(1-\alpha)} \leq 1 \tag{20}$$

If

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{D_n(k, A, B, \alpha, \beta, \gamma, \lambda)}{\beta\gamma(B-A)(1-\alpha)}$$

or if

$$|z| < \left[\frac{D_n(k, A, B, \alpha, \beta, \gamma, \lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)k(c+k)} \right]^{\frac{1}{k-1}} \tag{21}$$

then

$$\begin{aligned} |f'(z)-1| &= \left|-\sum_{k=2}^{\infty} k\frac{c+k}{c+1}|a_k|z^{k-1}\right| \le \sum_{k=2}^{\infty} k\frac{c+k}{c+1}|a_k||z|^{k-1} < \\ &< \sum_{k=2}^{\infty}\frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)|a_k|}{\beta\gamma(B-A)(1-\alpha)} \le 1 \end{aligned}$$

But from $|f'(z)-1|<1, |z|<R$, we deduce that f is univalent in the disc $|z|<R$.

The result is sharp and the extremal function is given by (18). □

Theorem 11. *Let* $c\in\mathbb{R}$, $c>-1$. *If*

$$F(z) = z - \sum_{k=2}^{\infty}|a_k|z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$$

then the function $f(z)$ *given by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt$$

is starlike of order p $(0\le p<1)$ *in* $|z|<R^*(p,A,B,\alpha,\beta,\gamma)$ *where*

$$R^* = \inf_k\left[\frac{(1-p)(c+1)D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad ; \quad k\ge 2.$$

The result is sharp.

Proof. Is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right|<(1-p)$, in $|z|<R^*$.

Now

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}|a_k|z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{c+k}{c+1}|a_k|z^{k-1}}\right| \le \frac{\sum_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}\frac{c+k}{c+1}|a_k||z|^{k-1}} < 1-p$$

provided

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)|a_k||z|^{k-1}<1$$

By using

$$\sum_{k=2}^{\infty}\frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)|a_k|}{\beta\gamma(B-A)(1-\alpha)}\le 1$$

the inequality

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)|a_k||z|^{k-1}<1$$

holds if

$$\frac{k-p}{1-p}\frac{c+k}{c+1}|z|^{k-1} < \frac{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \quad ; \quad k\ge 2$$

or if

$$|z| < \left[\frac{(1-p)(c+1)D_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad ; \quad k\ge 2.$$

Hence, $f(z) \in S^*(p)$ in $|z| < R^*$. The sharpness follows if we take the function $F(z)$, given by

$$F(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)z^k}{D_n(k,A,B,\alpha,\beta,\gamma,\lambda)},\ k \geq 2.$$

□

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