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BIVARIATE SPLINE-POLYNOMIAL INTERPOLATION

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Let $\Delta \subseteq \mathbb{R}^2$ be an arbitrary domain, f a real-valued function defined on Δ, $Z = \{z_i|\ z_i = (x_i, y_i),\ i = \overline{1,N}\} \subset \Delta$ and $I(f) = \{\lambda_k f|\ k = 1,\dots,N\}$ a set of informations about f (evaluations of f and of certain of its derivatives at $z_1,\dots,z_N$).

A general interpolation problem is: for a given function f find a function g that interpolates the data $I(f)$ i.e.

$$\lambda_k f = \lambda_k g, \quad k = \overline{1,N}.$$

A solution of such a problem can be obtain by the generalization of the bivariate Lagrange formula for the rectangular grid $\Pi = \{x_0,\dots,x_m\} \times \{y_0,\dots,y_n\}$:

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v(y)}{(y-y_j)v'(y_j)} f(x_i,y_j) + (R_{mn}f)(x,y) \qquad (1)$$

where

$$(R_{mn}f)(x,y) = u(x)[x,x_0,\dots,x_m; f(\cdot,y)] + \sum_{i=0}^{m} \frac{u(x)v(y)}{(x-x_i)u'(x_i)}[y,y_0,\dots,y_n; f(x_i,\cdot)]$$

with $u(x) = (x-x_0)\dots(x-x_m)$ and $v(y) = (y-y_0)\dots(y-y_n)$.

A first generalization of the formula (1) was given by J.F. Steffensen [4]:

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_j)v_i'(y_j)} + (R_{m,n_i}f)(x,y) \qquad (2)$$

where

$$(R_{m,n_i}f)(x,y) = u(x)[x,x_0,\dots,x_m; f(\cdot,y)] + \sum_{i=0}^{m} \frac{u(x)v_i(y)}{(x-x_i)u'(x_i)}[y,y_0,\dots,y_{in_i}; f(x_i,\cdot)]$$

with

$$v_i(y) = (y-y_0)\dots(y-y_{n_i}).$$

The interpolation grid here is $\Pi_1 = \{(x_i,y_{ij})|\ i = \overline{0,m},\ j = \overline{0,n_i}\}$.

A second generalization of the Lagrange interpolation formula (1), that is also an extension of the Steffensen formula (2) was given by D.D. Stancu [2]:

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i,y_{ij}) + \\ +(R_{m,n_i}f)(x,y) \qquad (3)$$

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where

$$(R_{m,n_i}f)(x,y) = u(x)[x,x_0,\dots,x_m;f(\cdot,y)] + \sum_{i=0}^{m} \frac{u(x)v_i(y)}{(x-x_i)u'(x_i)}[y,y_{i0},\dots,y_{in_i};f(x,\cdot)]$$

with $v_i(y) = (y-y_{i0})\dots(y-y_{in_i})$ and the interpolation set

$$\Pi_2 = \{(x_i,y_{ij})|\ i=\overline{0,m},\ j=\overline{0,n_i}\}.$$

Remark 1. The Steffensen formula (3) does not solve the general interpolation problem, Π_1 is only a particular case of the interpolatory set $\{z_1,\dots,z_N\}$.

Remark 2. Formula (3) is really a solution of the considered general problem. Indeed, let $Z_k \subset Z$ be the set of nodes (x_i,y_i), $i=\overline{1,N}$ with the same abscises x_k, i.e. $Z_k = \{(x_k,y_{kj})|\ j=\overline{0,n_k}\}$ for all $k=0,1,\dots,m$. We have $Z_i \neq Z_j$ for $i \neq j$ and $Z = Z_0 \cup \dots \cup Z_m$.

If L_m^x is the Lagrange's operator for the interpolates nodes $x_0,\dots,x_m$ and $L_{n_i}^y$, $i=\overline{0,m}$ are the Lagrange's operators for the nodes $y_{i0},\dots,y_{in_i}$ respectively, then we have

$$f = L_m^x f + R_m^x f \tag{4}$$

with

$$(L_m^x f)(x,y) = \sum_{i=0}^{m} \frac{u(x)}{(x-x_i)u'(x_i)} f(x_i,y)$$

and

$$f(x_i,\cdot) = (L_{n_i}^y f)(x_i,\cdot) + (R_{n_i}^y f)(x_i,\cdot), \quad i=\overline{0,m} \tag{5}$$

with

$$(L_{n_i}^y f)(x_i,y) = \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i,y_{ij}).$$

If the remainder terms are written with the divided differences, from (4) and (5) follows formula (3).

Remark 3. Usually the degree m of the operator L_m^x is more greater than the largest degree of $L_{n_i}^y$ i.e. $m \gg \max\{n_0,\dots,n_m\}$, which imply a large computational complexity of the polynomial interpolation from (3), say

$$(Pf)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_{ij})v'(y_{ij})} f(x_i,y_{ij}).$$

From this reason and the another ones, instead of Lagrange polynomial operator L_m^x we can use a spline interpolation function of Lagrange, Hermite or Birkhoff type.

1. **Spline polynomial interpolation of Lagrange type**

Let $S_{L,2n-1}^x$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f with regard to the variable x at the nodes (x_k,y), $k=\overline{0,m}$ i.e.

$$(S_{L,2n-1}^x f)(x,y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=0}^{m} b_j (x-x_j)_+^{2n-1} \tag{6}$$

for which

$$\begin{cases} (S^x_{L,2n-1}f)(x_k,y) = f(x_k,y), & k=\overline{0,m} \\ (S^x_{L,2n-1}f)^{(p,0)}(\alpha,y) = 0, & p=\overline{n,2n-1},\ \alpha > x_m \end{cases} \tag{7}$$

The spline function of Lagrange type can also be written in the form

$$(S^x_{L,2n-1}f)(x,y) = \sum_{k=0}^{m} s_k(x) f(x_k,y)$$

where s_k are the corresponding cardinal splines i.e., they are of the same form (6), but with the interpolatory conditions

$$s_k(x_j) = \delta_{kj}, \quad k,j=\overline{0,m}.$$

This way, formula (3) becomes

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} s_i(x) \frac{v_i(y)}{(y-y_{ij})v'(y_{ij})} f(x_i,y_{ij}) + (Rf)(x,y) \tag{8}$$

where $(Rf)(x,y)$ is the remainder term.

Taking into account that for $f(\cdot,y) \in C^n[x_0,x_m]$

$$(R^x_{L,2n-1}f)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds$$

with

$$\varphi_n(x,s) = R^x\left[\frac{(x-s)_+^{n-1}}{(n-1)!}\right]$$

it follows

Theorem 1. If $f \in C^{n,0}(\Delta)$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds + \tag{9}$$

$$+\sum_{i=0}^{m} s_i(x) v_i(y) [y, y_{i0}, \ldots, y_{in_i}; f(x_i,\cdot)]$$

and if $f \in C^{n,p+1}(\Delta)$ with $p = \max\{n_0,\ldots,n_m\}$ we have

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_n(x,s) f^{(n,0)}(s,y)ds + \tag{10}$$

$$+\sum_{i=0}^{m} s_i(x) \int_{y_{i0}}^{y_{in_i}} \psi_{n_i}(y,t) f^{(0,n_i+1)}(x_i,t)$$

with

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} \frac{(y_{ij}-t)_+^{n_i}}{n_i!}.$$

2. Spline polynomial interpolation of Hermite type

Let $S^x_{H,2n-1}$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f and certain of its derivatives with regard to the variable x at the nodes (x_k, y), $k=\overline{0,m}$, i.e.

$$(S^x_{H,2n-1}f)(x,y)=\sum_{i=0}^{n-1}a_ix^i+\sum_{k=0}^{m}\sum_{j=0}^{q_k}b_{kj}(x-x_k)_+^{2n-j-1} \tag{11}$$

for which

$$\begin{cases} (S^x_{H,2n-1}f)^{(j,0)}(x_k,y)=f^{(j,0)}(x_k,y), & k=\overline{0,m},\ j=\overline{0,q_k} \\ (S^x_{H,2n-1}f)^{(p,0)}(\alpha,y)=0, & p=\overline{n,2n-1},\ \alpha>x_m \end{cases} \tag{12}$$

The spline function of Hermite type can also be written in the form

$$(S^x_{H,2n-1}f)(x,y)=\sum_{k=0}^{m}\sum_{j=0}^{q_k}s_{kj}(x)f^{(j,0)}(x_k,y)$$

where s_{kj} are the corresponding cardinal splines i.e., they are of the same form (11), but with the interpolatory conditions

$$\begin{cases} s^{(q)}_{kj}(x_\nu)=0, & k=\overline{0,m},\ \nu\neq k,\ q=\overline{0,q_\nu} \\ s^{(q)}_{kj}(x_k)=\delta_{jq}, & q=\overline{0,q_k} \\ s^{(p)}_{kj}(\alpha)=0, & p=\overline{n,2n-1},\ \alpha>x_m \end{cases}$$

This way, formula (3) becomes

$$f(x,y)=\sum_{i=0}^{m}\sum_{l=0}^{q_i}s_{il}(x)\sum_{j=0}^{n_i}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}f^{(l,0)}(x_i,y_{ij})+(Rf)(x,y) \tag{13}$$

where

$$v_i(y)=(y-y_{i0})\dots(y-y_{in_i})$$

and $(Rf)(x,y)$ is the remainder term.

In this case the set of information about f is

$$I(f)=\{f^{(l,0)}(x_i,y_{ij})|\ i=\overline{0,m},\ j=\overline{0,n_i},\ l=\overline{0,q_i}\} \tag{14}$$

Taking into account that for $f(\cdot,y)\in C^n[x_0,x_m]$

$$(R^x_{H,2n-1}f)(x,y)=\int_{x_0}^{x_m}\varphi_H(x,s)f^{(n,0)}(s,y)ds$$

with

$$\varphi_H(x,s)=\frac{(x-s)_+^{n-1}}{(n-1)!}-\sum_{i=0}^{m}\sum_{l=0}^{q_i}s_{il}(x)\frac{(x_i-s)_+^{n-l-1}}{(n-l-1)!}$$

and that

$$(Rf)(x,y)=(R^x_{H,2n-1}f)(x,y)+\sum_{i=0}^{m}\sum_{l=0}^{q_i}s_{il}(x)(R^y_{L,n_i}f^{(l,0)})(x_i,y)$$

it follows

Theorem 2. If $f \in C^{n,0}(\Delta)$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_H(x,s) f^{(n,0)}(s,y)ds+ \tag{15}$$

$$+\sum_{i=0}^{m}\sum_{l=0}^{q_i} s_{il}(x) v_i(y)[y, y_{i0}, \ldots, y_{in_i}; f^{(l,0)}(x_i, \cdot)]$$

and if $f \in C^{n,p+1}(\Delta)$ with $p = \max\{n_0, \ldots, n_m\}$ then

$$(Rf)(x,y) = \int_{x_0}^{x_m} \varphi_H(x,s) f^{(n,0)}(s,y)ds+ \tag{16}$$

$$+\sum_{i=0}^{m}\sum_{l=0}^{q_i} s_{il}(x) \int_{y_{i0}}^{y_{i,n_i}} \psi_{n_i}(y,t) f^{(l,n_i+1)}(x_i,t)dt$$

with

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} \frac{(y_{ij}-t)_+^{n_i}}{n_i!}$$

3. Spline polynomial interpolation of Birkhoff type

Let $S^x_{B,2n-1}$ be the spline interpolation operator of the degree $2n-1$, that interpolates the function f and certain of its partial derivatives to the variable x at the nodes (x_k, y), $k = \overline{0,m}$, i.e.

$$(S^x_{B,2n-1}f)(x,y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{k=0}^{m}\sum_{j \in I_k} b_{kj}(x-x_k)_+^{2n-j-1} \tag{17}$$

for which

$$\begin{cases} (S^x_{B,2n-1}f)^{(j,0)}(x_k,y) = f^{(j,0)}(x_k,y), & k=\overline{0,m},\ j \in I_k \\ (S^x_{B,2n-1}f)^{(p,0)}(\alpha,0) = 0, & p = \overline{n,2n-1},\ \alpha > x_m \end{cases} \tag{18}$$

The spline function of Birkhoff type can also be written in the form

$$(S^x_{B,2n-1}f)(x,y) = \sum_{k=0}^{m}\sum_{j \in I_k} s_{kj}(x) f^{(j,0)}(x_k,y)$$

where s_{kj} are the corresponding cardinal splines i.e., they are of the same form (17), but with the interpolatory conditions:

$$\begin{cases} s_{kj}^{(q)}(x_\nu) = 0, & k=\overline{0,m},\ \nu \neq k,\ q \in I_\nu \\ s_{kj}^{(q)}(x_k) = \delta_{jq}, & q \in I_k \\ s_{kj}^{(p)}(\alpha) = 0, & p = \overline{n,2n-1},\ \alpha > x_m \end{cases}$$

If the set of informations of f is

$$I(f) = \{f^{(l,0)}(x_i, y_{ij}) |\ i = \overline{0,m},\ j = \overline{0,n_i},\ l \in I_i\} \tag{19}$$

we can use the interpolation formula of Lagrange

$$f^{(l,0)}(x_i,y) = (L^y_{n_i} f^{(l,0)})(x_i,y) + (R^y_{L,n_i} f^{(l,0)})(x_i,y) \tag{20}$$

This way, formula (3) becomes

$$f(x,y)=\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)\sum_{j=0}^{n_i}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}f^{(l,0)}(x_i,y_{ij})+(Rf)(x,y) \tag{21}$$

where $(Rf)(x,y)$ is the remainder term.

Taking into account that for $f(\cdot,y)\in C^n[x_0,x_m]$

$$(R^x_{B,2n-1}f)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)ds$$

with

$$\varphi_B(x,s)=\frac{(x-s)_+^{n-1}}{(n-1)!}-\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)\frac{(x_i-s)_+^{n-l-1}}{(n-l-1)!}$$

and that

$$(Rf)(x,y)=(R^x_{B,2n-1}f)(x,y)+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)(R^y_{L,n_i}f^{(l,0)})(x_i,y)$$

it follows

Theorem 3. If $f\in C^{n,0}(\Delta)$ then

$$(Rf)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)+ \tag{22}$$

$$+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)v_i(y)[y,y_{i0},\dots,y_{in_i};f^{(l,0)}(x_i,\cdot)]$$

with $v_i(y)=(y-y_{i0})\dots(y-y_{in_i})$ and if $f\in C^{n,p+1}(\Delta)$ with $p=\max\{n_0,\dots,n_m\}$ then

$$(Rf)(x,y)=\int_{x_0}^{x_m}\varphi_B(x,s)f^{(n,0)}(s,y)ds+ \tag{23}$$

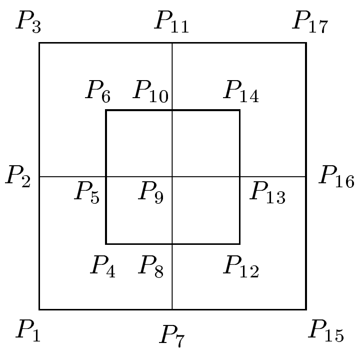
$$+\sum_{i=0}^{m}\sum_{l\in I_i}s_{il}(x)v_i(y)\int_{y_{i0}}^{y_{in_i}}\psi_{n_i}(y,t)f^{(l,n_i+1)}(x_i,t)dt$$

where

$$\psi_{n_i}(y,t)=\frac{(y-t)_+^{n_i}}{n_i!}-\sum_{j=0}^{n_i}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}\frac{(y_{ij}-t)_+^{n_i}}{n_i!}$$

4. **Example**

One considers the function $f(x,y)=\exp(-x^2-y^2)$ on the rectangular domains $\Delta=[-1,1]\times[-1,1]$ and the interpolation nodes P_1-P_{17}



$P_1 = (-1,-1)$
$P_2 = (-1,0)$
$P_3 = (-1,1)$
$P_4 = (-0.5,-0.5)$
$P_5 = (-0.5,0)$
$P_6 = (-0.5,0.5)$
$P_7 = (0,-1)$
$P_8 = (0,-0.5)$
$P_9 = (0,0)$
$P_{10} = (0,0.5)$
$P_{11} = (0,1)$
$P_{12} = (0.5,-0.5)$
$P_{13} = (0.5,0)$
$P_{14} = (0.5,0.5)$
$P_{15} = (1,-1)$
$P_{16} = (1,0)$
$P_{17} = (1,1)$

We will use the formulas (8), (13) and (21) for $n = 2$ (cubic spline with regard the variable x).

In fig. 1 is given the graph for the function f.

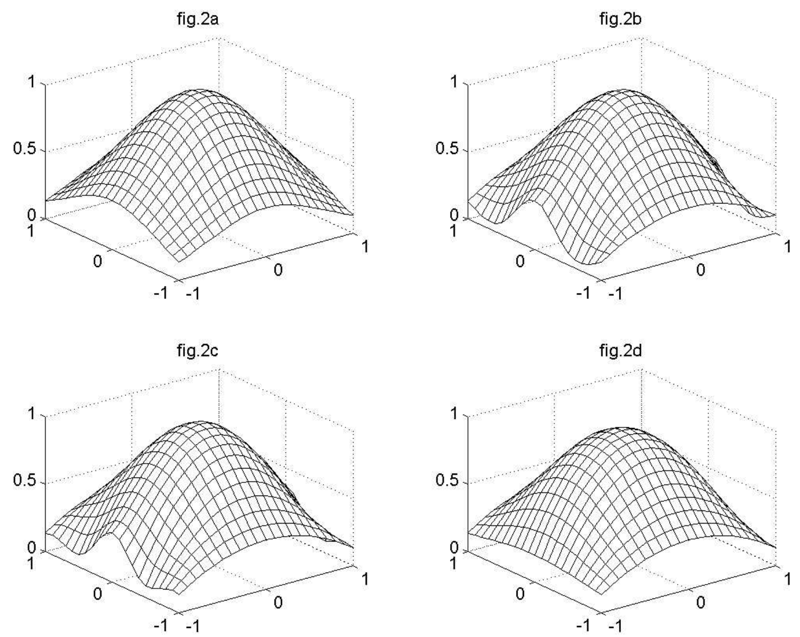
In fig. 2 is used the information of Lagrange type.

In fig. 3 is used information of Hermite type, i.e.

$$\{f^{(j,0)}(P_i):\ i = \overline{1,17},\ j = 0,1\}$$

In fig. 4 is used a set of information of Birkhoff type:

$$\{f(P_i):\ i = 1,2,3,15,16,17\} \cup \{f^{(1,0)}(P_i):\ i = \overline{4,14}\}.$$



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