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A CLASS OF EVEN DEGREE SPLINES OBTAINED THROUGH A MINIMUM CONDITION

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. A class of splines minimizing a special functional is investigated. This class is determined by the solution of quadratic programming problem. Convergence results and some numerical examples are given.

1. **Introduction**

The construction of splines, verifying minimum conditions has been proposed among others in [2], [5], [6]. In such papers the splines are interpolating the approximated function in the nodes and while in [5] the constructive method can be applied, in theory, for a spline of an arbitrary degree *m,* minimizing the integral $\int_I [g'(x)]^2 dx$, in [2] e [6] a cubic polynomial interpolating splines are considered satisfying some minimum conditions.

In particular, in [6], the considered splines have been applied for constructing quadrature sums approximating the Cauchy principal value integrals

$$I(wf;t) = \fint_{-1}^{1} w(x) \frac{f(x)}{x-t} dx. \tag{1.1}$$

In this paper, utilizing the method proposed in [2], we construct the spline of even degree minimizing the functional

$$F(f) := \int_I [f^{(3)}(x)]^2 dx \qquad\qquad f \in W_2^3(I) \tag{1.2}$$

where, denoting $AC(I)$ the set of absolute continuous functions on I,

$$W_2^3(I) := \left\{ f : I \to I\!R, \quad f^{(0)} \in AC(I) \quad and \quad f^{(3)} \in L_2(I) \right\} . \tag{1.3}$$

This class of splines, called *interpolating-derivative* splines of degree $2m$, $m \geqslant 2$, has been determined in [3] by solving a linear system of $m+n+1$ equations, where n is the number of internal knots of the partition, and then the authors proved that the constructed spline solves problem (1.2). The convergence is proved by supposing $f \in W_2^{m+1}$.

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In this paper, exploiting the different form that we use for defining the interpolating - derivative spline, we can obtain convergence results under weaker conditions on f, that gives more flexibility in the applications, as for example, when we consider the numerical evaluation of Cauchy singular integrals [8].

In Section 2 we give the details of the construction of the interpolating-derivative spline. In Section 3 we give some convergence results. Finally, in Section 4, some numerical experiments on test functions f are reported. In Appendix we prove some propositions whose results are necessary for proving theorem 2.5 and proposition 2.7 in Section 2 and theorems 3.2, 3.3 in Section 3.

2. **Construction of derivative-interpolating spline**

Let m, $n \geqslant m$ two given integer positive numbers and $Y \in I\!R^{n+1}, Y := \{y_0, y_1', ..., y_n'\}$ a given vector and

$$\Delta_n := \{a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b\}$$

a given partition of $I \equiv [a, b]$ in $n+1$ subintervals $I_k := [x_k, x_{k+1})$, $k = 0, 1, ..., n$, limiting ourselves, for the sake of simplicity, to consider an uniform partition Δ_n, with $h = x_{i+1} - x_i$, $i = 0, 1, ..., n$.

We denote by $I\!P_k$ the set of polynomials of degree $\leq k$. Consider the space of polynomial splines of degree $2m$

$$S_{2m}(\Delta_n) = \left\{ \begin{array}{l} s : s(x) = s_i(x) \in I\!P_{2m},\ x \in I_i,\ i = 0, 1, ..., n; \\ D^j s_{i-1}(x_i) = D^j s_i(x_i),\ j = 0, 1, ..., 2m-1,\ i = 1, 2, ..., n \end{array} \right\} \tag{2.1}$$

with simple knots $x_1, x_2, ..., x_n$. The space $S_{2m}(\Delta_n) \subset C^{2m-1}(I)$.

A function $s_f \in S_{2m}(\Delta_n)$ is called *derivative-interpolating* if

$$s_f(x_0) = y_0,\ s_f'(x_i) = y_i',\ i = 1, 2, ..., n;\ y_0 = f(x_0),\ y_i' = f(x_i). \tag{2.2}$$

Limiting ourselves to consider $m = 2$, if we set

$$M_i = s_f^{(2m-1)}(x_i),\ i = 0, 1, ..., n+1,$$

by successive integrations, we obtain

$$\begin{array}{rl} s_f(x)|_{I_i} = & [M_{i+1}(x - x_i)^4 - M_i(x - x_{i+1})^4]/(4!h)+ \\ & +a_i(x - x_i)^2/2 + b_i(x - x_i) + c_i, \quad i = 0, 1, ..., n. \end{array} \tag{2.3}$$

By imposing the conditions (2.1) and (2.2), we obtain

$$\begin{cases} a_i &= \frac{y'_{i+1}-y'_i}{h} - \frac{h}{6}(M_{i+1}-M_i) & i=1,...,n-1 \\ b_0 &= y'_1 - \frac{h^2}{6}M_1 - a_0 h & \\ b_i &= y'_i - \frac{h^2}{6}M_i & i=1,...,n \\ c_0 &= y_0 + \frac{h^3}{24}M_0 & \\ c_1 &= c_0 + y'_1 h - \frac{h^3}{12}M_1 - \frac{h^2}{2}a_0 & \\ c_i &= c_{i-1} + (y'_i + y'_{i-1})\frac{h}{2} - \frac{h^3}{12}M_{i-1} & i=2,...,n \\ M_i h &= a_i - a_{i-1} & i=1,...,n. \end{cases} \tag{2.4}$$

Substituting the first equations of (2.4) in the last ones, we obtain a linear system

$$\widetilde{A}\widetilde{M} = \underline{b}^*(a_0, a_n) \tag{2.5}$$

where

$$\widetilde{A} = \begin{bmatrix} 5 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 5 \end{bmatrix}, \quad \widetilde{M} = \begin{bmatrix} M_1 \\ . \\ . \\ . \\ M_n \end{bmatrix},$$

$$\underline{b}^* = \frac{6}{h}\left[\frac{y'_2 - y'_1}{h} - a_0, \dots, \frac{y'_{i+1} - y'_i}{h} - \frac{y'_i - y'_{i-1}}{h}, \dots, -\frac{y'_n - y'_{n-1}}{h} + a_n\right]^T.$$

The spline function $s_f(x)$ will be determined by solving the following problem

$$\begin{cases} \min M^T \bar{A} M \\ \widetilde{A}\widetilde{M} = \underline{b}^*(a_0, a_n) \end{cases} \tag{2.6}$$

with $M = [M_{0,\dots,}M_{n+1}]^T$,

$$\bar{A} = \left[\begin{array}{c|ccccc|c} 2 & 1 & & & & & 0 \\ \hline 1 & 4 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & 4 & 1 & \\ \hline 0 & & & & 1 & 2 & \end{array}\right] = \left[\begin{array}{c|c|c} 2 & \underline{e}_1^T & 0 \\ \hline \underline{e}_1 & A^* & \underline{e}_n \\ \hline 0 & \underline{e}_n^T & 2 \end{array}\right], \tag{2.7}$$

where

$$A^* = \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix} \tag{2.8}$$

and $\underline{e}_1$, $\underline{e}_n$ are the vectors $[1,0,...,0]^T$, $[0,0,...,1]^T$ respectively.

We can write $\underline{b}^* = \underline{b} - \underline{e}_1\widetilde{a}_0 + \underline{e}_n\widetilde{a}_n$ with

$$\underline{b} = \frac{6}{h}\left[\frac{y_2' - y_1'}{h}, \ldots, \frac{y_{i+1}' - y_i'}{h} - \frac{y_i' - y_{i-1}'}{h}, \ldots, -\frac{y_n' - y_{n-1}'}{h}\right]^T \tag{2.9}$$

and $\widetilde{a}_0 = \frac{6}{h}a_0$, $\widetilde{a}_n = \frac{6}{h}a_n$.

Considering that $\widetilde{A}$ is a symmetric positive definite and then, non singular matrix, from (2.5) we get

$$\widetilde{M} = \widetilde{A}^{-1}(\underline{b} - \underline{e}_1\widetilde{a}_0 + \underline{e}_n\widetilde{a}_n) \tag{2.10}$$

thus:

$$\min M^T \bar{A} M = \min\left\{\left[M_0 \widetilde{M}^T M_{n+1}\right]\begin{bmatrix} 2 & \underline{e}_1^T & 0 \\ \underline{e}_1 & A^* & \underline{e}_n \\ 0 & \underline{e}_n^T & 2 \end{bmatrix}\left[M_0 \widetilde{M}^T M_{n+1}\right]^T\right\}. \tag{2.11}$$

Using (2.10), the problem amounts to find out firstly the vector

$$N = [\widetilde{a}_0, -\widetilde{a}_n, -M_0, -M_{n+1}]^T,$$

solution of the linear system

$$BN = P \tag{2.12}$$

where, by setting $C = \widetilde{A}^{-1}A^*\widetilde{A}^{-1}$,

$$B = \begin{bmatrix} B_1 & B_2 \\ B_2 & 2I_2 \end{bmatrix}, B_1 = \begin{bmatrix} \underline{e}_1^T C \underline{e}_1 & \underline{e}_1^T C \underline{e}_n \\ \underline{e}_n^T C \underline{e}_1 & \underline{e}_n^T C \underline{e}_n \end{bmatrix}, B_2 = \begin{bmatrix} \underline{e}_1^T \widetilde{A}^{-1}\underline{e}_1 & \underline{e}_1^T \widetilde{A}^{-1}\underline{e}_n \\ \underline{e}_n^T \widetilde{A}^{-1}\underline{e}_1 & \underline{e}_n^T \widetilde{A}^{-1}\underline{e}_n \end{bmatrix}. \tag{2.13}$$

I_2 is the second order identity matrix and

$$P = \left[\underline{e}_1^T C\underline{b},\ \underline{e}_n^T C\underline{b},\ \underline{e}_1^T\widetilde{A}^{-1}\underline{b},\ \underline{e}_n^T\widetilde{A}^{-1}\underline{b}\right]^T. \tag{2.14}$$

Once determined N, we shall determine $s_f(x)$ by solving the system (2.5).

Before proving the below theorem 2.5, we need to investigate some properties of matrices $\widetilde{A}$, $\widetilde{A}^{-1}$and C.

Proposition 2.1. *The matrix* $\widetilde{A} = (a_{ij})_{i,j=1}^n$, *is:*

(a) *symmetric, positive definite;*
(b) *persymmetric, i.e.* $a_{ij} = a_{n-i+1,n-j+1}$, $i,j = 1,...,n$;
(c) *totally positive (T.P.), i.e. all the minors are* ≥ 0;
(d) *oscillatory, then all the eigenvalues of* $\widetilde{A}$ *are distinct, real and positive.*

Proof. It is straightforward to verify (a), (b), (c). The property (d) follows by considering that a non singular T.P. matrix having the entries $a_{ik} \neq 0$, $|i-k| \leq 1$ is oscillatory [4]. □

Proposition 2.2. *The infinitive norm of* $\widetilde{A}^{-1}$ *satisfies the following relation*

$$\frac{1}{6} \leq \left\| \widetilde{A}^{-1} \right\|_{\infty} \leq \frac{1}{2}. \tag{2.15}$$

Proof. (For the proof, see Appendix). □

Proposition 2.3. *The entries* a_{1j}^{-1}, $j = 1, ..., n$ *of* $\widetilde{A}^{-1}$ *have decreasing absolute values, the sign of* $(-1)^{j-1}$*, in particular, the following inequalities:*

$$\frac{1}{5} \leq a_{11}^{-1} = \underline{e}_1^T \widetilde{A}^{-1} \underline{e}_1 \leq \frac{1}{4}, \tag{2.16}$$

$$\left| a_{1n}^{-1} \right| = \left| \underline{e}_1^T \widetilde{A}^{-1} \underline{e}_n \right| \leq \begin{cases} \frac{1}{24} & \text{if} \quad n = 2, \\ \\ \frac{1}{90} & \text{if} \quad n \geq 3 \end{cases} \tag{2.17}$$

hold.

Proof. (For the proof, see Appendix). □

Proposition 2.4. *Let* $C = \widetilde{A}^{-1} A^* \widetilde{A}^{-1}$*. For the entries* $c_{11} = \underline{e}_1^T C \underline{e}_1$, $c_{1n} = \underline{e}_1^T C \underline{e}_n$ *we have:*

$$0 < c_{11} < 1 \tag{2.18}$$

$$|c_{1n}| < c_{11}. \tag{2.19}$$

Proof. (For the proof, see Appendix). □

Now we prove the following:

Theorem 2.5. *The system (2.12) is determined and the solution is*

$$N = \left[\left[\underline{e}_1^T \widetilde{A}^{-1} \underline{b},\ \underline{e}_n^T \widetilde{A}^{-1} \underline{b} \right] B_2^{-1},\ 0,\ 0 \right]^T. \tag{2.20}$$

Proof. Considering that $\underline{e}_1^T \widetilde{A}^{-1} \underline{e}_n = \underline{e}_n^T \widetilde{A}^{-1} \underline{e}_1$ and for the properties of $\widetilde{A}^{-1}$ and the definition of the symmetric positive matrix A^*, one has $\underline{e}_1^T C \underline{e}_n = \underline{e}_n^T C \underline{e}_1$, (2.11) can be written in the form

$$\begin{bmatrix} B_1 & B_2 \\ B_2 & 2I_2 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} \underline{t}_1 \\ \underline{t}_2 \end{bmatrix}, \tag{2.21}$$

where

$\underline{x} = [\widetilde{a}_0, -\widetilde{a}_n]$, $\underline{y} = [-M_0, -M_{n+1}]$, $\underline{t}_1 = \left[\underline{e}_1^T C \underline{b},\ \underline{e}_n^T C \underline{b}\right]^T$, $\underline{t}_2 = \left[\underline{e}_1^T \widetilde{A}^{-1} \underline{b},\ \underline{e}_n^T \widetilde{A}^{-1} \underline{b} \right]^T$.

Since $A^* = \widetilde{A} - \left(\underline{e}_1 \underline{e}_1^T + \underline{e}_n \underline{e}_n^T \right)$, and then, $\widetilde{A}^{-1} A^* \widetilde{A}^{-1} = \widetilde{A}^{-1} - \widetilde{A}^{-1} \left(\underline{e}_1 \underline{e}_1^T + \underline{e}_n \underline{e}_n^T \right) \widetilde{A}^{-1}$ there results

$$B_1 = B_2 \left(I_2 - B_2 \right) \tag{2.22}$$

$$\underline{t}_1 = \left(I_2 - B_2 \right) \underline{t}_2, \tag{2.23}$$

and the system (2.21) reduces to:

$$\begin{bmatrix} B_2 \left(I_2 - B_2 \right) & B_2 \\ B_2 & 2I_2 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} \left(I_2 - B_2 \right) \underline{t}_2 \\ \underline{t}_2 \end{bmatrix}. \tag{2.24}$$

The system (2.21) has unique solution.

In fact [4], since

$$2B_2I_2 = 2I_2B_2, \det \begin{bmatrix} B_2\,(I_2 - B_2) & B_2 \\ B_2 & 2I_2 \end{bmatrix} = \det(B_2)\det(2I_2 - 3B_2)$$

and using the results of propositions 2.2 and 2.3, it is straightforward to verify that $\det(B_2) \neq 0$ and $\det(2I_2 - 3B_2) \neq 0$. From the second equation of (2.24) we obtain $\underline{x} = B_2^{-1}(\underline{t}_2 - 2I_2\underline{y})$, and, substituting in the first one, there results:

$$B_2\,(I_2 - B_2)\,B_2^{-1}(\underline{t}_2 - 2\underline{y}) + B_2\underline{y} = (I_2 - B_2)\,\underline{t}_2,$$

that implicates

$$(3B_2 - 2I_2)\underline{y} = \underline{0}. \tag{2.25}$$

Therefore we obtain the solution $\underline{y} = 0$, $\underline{x} = B_2^{-1}\underline{t}_2$ and theorem 2.5 is proved. □

Corollary 2.6. *For the spline $s_f(x)$ the following property*

$$M_1 = s_f^{(2m-1)}(x_1) = M_n = s_f^{(2m-1)}(x_n) = 0 \tag{2.26}$$

holds.

Proof. Taking into account that, from the relation $B_2\underline{x} + 2\underline{y} = \underline{t}_2$, we obtain

$$\begin{bmatrix} M_0 \\ M_{n+1} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1}\underline{b} \\ \underline{e}_n^T\widetilde{A}^{-1}\underline{b} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1}\underline{e}_1 & -\underline{e}_1^T\widetilde{A}^{-1}\underline{e}_n \\ \underline{e}_n^T\widetilde{A}^{-1}\underline{e}_1 & -\underline{e}_n^T\widetilde{A}^{-1}\underline{e}_n \end{bmatrix}\begin{bmatrix} \widetilde{a}_0 \\ \widetilde{a}_n \end{bmatrix}, \tag{2.27}$$

we get the thesis considering the first and the last equation in (2.10), that is:

$$\begin{bmatrix} M_1 \\ M_n \end{bmatrix} = \begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1}\underline{b} \\ \underline{e}_n^T\widetilde{A}^{-1}\underline{b} \end{bmatrix} - \begin{bmatrix} \underline{e}_1^T\widetilde{A}^{-1}\underline{e}_1 & -\underline{e}_1^T\widetilde{A}^{-1}\underline{e}_n \\ \underline{e}_n^T\widetilde{A}^{-1}\underline{e}_1 & -\underline{e}_n^T\widetilde{A}^{-1}\underline{e}_n \end{bmatrix}\begin{bmatrix} \widetilde{a}_0 \\ \widetilde{a}_n \end{bmatrix}. \tag{2.28}$$

□

From theorem 2.5 and corollary 2.6 we can deduce that, as aspected, the obtained spline $s_f(x)$ reduces to a polynomial of second degree in the subintervals I_0 and I_n.

Remark 2.1. *In* [3, theorem 2] *the construction of such spline is obtained by solving a linear system of $m + n + 1$ equations that can have an increasing condition number when n increases. Our method is based on the solution of two linear systems, having matrix $\widetilde{A}$ and B respectively. By proposition 2.2, for each n, for the condition number of $\widetilde{A}$, we have $K_\infty\left(\widetilde{A}\right) \leq 3$; now we prove the following*

Proposition 2.7. *For the condition number $K_\infty(B)$ the inequality*

$$K_\infty(B) \leq \frac{4477}{219} \simeq 20.44, \ \textit{if}\ n \geq 3, \tag{2.29}$$

holds.

Proof. $B = \begin{bmatrix} B_1 & B_2 \\ B_2 & 2I_2 \end{bmatrix}$, then

$$B^{-1} = \begin{bmatrix} (2I_2 - 3B_2)^{-1} & 0 \\ 0 & (2I_2 - 3B_2)^{-1} \end{bmatrix} \begin{bmatrix} 2B_2^{-1} & -I_2 \\ -I_2 & I_2 - B_2 \end{bmatrix}.$$

Let $n \geq 3$. Using the results obtained in the propositions 2.3 and 2.4, one obtains

$$\|B\|_\infty \leq 2 + a_{11}^{-1} + \left|a_{1n}^{-1}\right| \leq \frac{407}{180}, \tag{2.30}$$

$$\|B_2\|_\infty = a_{11}^{-1} + \left|a_{1n}^{-1}\right| \leq \frac{47}{180} \tag{2.31}$$

and

$$\left\|B_2^{-1}\right\|_\infty = \frac{1}{\left|a_{11}^{-1}\right| - \left|a_{1n}^{-1}\right|} \leq 5.$$

Besides:

$$\left\|B^{-1}\right\|_\infty \leq \left\|(2I_2 - 3B_2)^{-1}\right\|_\infty \left(2\left\|B_2^{-1}\right\|_\infty + 1\right). \tag{2.32}$$

Using the results in [7], since for all vector $\underline{x}$ such that $\|\underline{x}\|_\infty = 1$, one has

$$\|(2I_2 - 3B_2)\,\underline{x}\|_\infty \geq 2 - 3\frac{47}{180} = \frac{73}{60}, \tag{2.33}$$

and then

$$\left\|B^{-1}\right\|_\infty \leq \frac{660}{73}. \tag{2.34}$$

Therefore

$$K_\infty(B) \leq \frac{407}{180}\frac{660}{73} \simeq 20.44, \text{ if } n \geq 3$$

and we get the thesis. □

3. **Convergence results**

Consider $I = [a, b]$ and the set W_2^3. In [3] has been proved the following

Theorem 3.1. *Let $f \in W_2^3(I)$ and let s_f be the derivative-interpolating spline, then*

$$\left\|f^{(k)} - s_f^{(k)}\right\|_\infty \leq \begin{cases} C_1 h^{2-\frac{1}{2}} \left\|f^{(3)}\right\|_2 & \text{if } k = 0, \\ C_2 h^{2-k+\frac{1}{2}} \left\|f^{(3)}\right\|_2 & \text{if } k = 1, 2, \end{cases} \tag{3.1}$$

where $C_1 = \sqrt{2}\,(b-a)$ and $C_2 = \sqrt{2}$.

We shall prove a new convergence theorem under weaker hypothesis on function f. For all $g \in C^1(I)$, we denote by

$$\omega(g'; h; I) = \max_{x, x+\delta \in I,\, 0 \leq \delta \leq h} |g'(x+\delta) - g'(x)|$$

the modulus of continuity of g'.

Supposing $f \in C^1(I)$, from (2.9), we obtain

$$\|\underline{b}\|_\infty \leq \frac{12}{h^2}\omega(f'; h; I) \tag{3.2}$$

then, using (2.15), (2.20):

$$\|a_0, a_n\|_\infty \leq \frac{5}{h}\omega\left(f'; h; I\right). \tag{3.3}$$

and consequently, from (2.10), we obtain:

$$\|M\|_\infty = \left\|\widetilde{M}\right\|_\infty \leq \frac{21}{h^2}\omega\left(f'; h; I\right). \tag{3.4}$$

If we consider that $f'(x_i) = y_i'$ and by (2.4):

$a_i \quad = \quad \frac{y_{i+1}' - y_i'}{h} - \frac{h}{6}(M_{i+1} - M_i) \quad i = 1, ..., n-1,$ we can write

$$|a_i| \leq \frac{8}{h}\omega\left(f'; h; I\right). \tag{3.5}$$

Therefore,

$$\|\underline{a}\|_\infty \leq \frac{8}{h}\omega\left(f'; h; I\right) \tag{3.6}$$

where $\underline{a} = [a_0, a_1, ..., a_n]^T$.

Theorem 3.2. *Let $f \in C^1(I)$ and $s_f(x)$ the interpolating-derivative spline quoted in Section 2 for a given partition Δ_n. Then*

$$\omega\left(s_f'; h; I\right) \leq C\omega\left(f'; h; I\right) \tag{3.7}$$

where C is a constant independent of h.

Proof. It suffices to show that for $\forall u, v \in I$, $u < v$:

$$\left|s_f'(v) - s_f'(u)\right| \leq \bar{C}\omega\left(f'; v-u; I\right).$$

Firstly consider $u, v \in [x_i, x_{i+1}]$, $i = 0, ..., n$; using the mean value theorem, we can derive:

$$\left|s_f'(v) - s_f'(u)\right| = \left|s_f''(\xi)\right| |v-u| \qquad \xi \in (u, v),$$

where $|v-u| \leq h$.

Since for any $\xi \in (u, v)$, from (3.3), (3.4), (3.5), there results

$\left|s_f''(\xi)\right| \leq \frac{29}{h}\omega\left(f'; h; I\right)$ if $u, v \in [x_i, x_{i+1}]\, t$, $i = 1, 2, ..., n-1$ and

$\left|s_f''(\xi)\right| \leq \frac{5}{h}\omega\left(f'; h; I\right)$ if $u, v \in [x_0, x_1]$ or $u, v \in [x_n, x_{n+1}]$,

recalling that [9], $\frac{|v-u|}{h}\omega\left(f'; h; I\right) \leq 2\omega\left(f'; |v-u|; I\right)$, we get

$$\left|s_f'(v) - s_f'(u)\right| \leq C_1\omega\left(f'; |v-u|; I\right), \qquad C_1 = 58. \tag{3.8}$$

If $u \in [x_i, x_{i+1}]$, $v \in [x_j, x_{j+1}]$, $i+1 \leq j$, then using (3.8) and the smoothness of modulus of continuity and, since being x_{i+1} and x_j internal nodes, $s_f'(x_{i+1}) = f'(x_{i+1})$, $s_f'(x_j) = f'(x_j)$:

$$\begin{aligned}\left|s_f'(v) - s_f'(u)\right| &\leq \left|s_f'(v) - s_f'(x_j)\right| + \left|s_f'(x_j) - s_f'(x_{i+1})\right| + \left|s_f'(x_{i+1}) - s_f'(u)\right| \\ &= \left|s_f'(v) - s_f'(x_j)\right| + \left|f'(x_j) - f'(x_{i+1})\right| + \left|s_f'(x_{i+1}) - s_f'(u)\right| \\ &\leq (2C_1 + 1)\,\omega\left(f'; |v-u|; I\right).\end{aligned}$$

This proves the theorem with $C = (2C_1 + 1)$. □

Supposing $f \in C^1(I)$, we define $r(x) = f(x) - s_f(x)$ and $r'(x) = f'(x) - s'_f(x)$, where $s_f(x)$ is the interpolating-derivative spline quoted in Section 2.

For $x \in I_i$, $i = 0, ..., n$ we can write

$$r'(x) = \begin{cases} r'(x_1) + (x - x_1)[x_1 x]\, r' & \text{if } x \in I_0, \\ r'(x_i) + (x - x_i)[x_i x]\, r' & \text{if } x \in I_i,\ i = 1, \dots, n. \end{cases} \tag{3.9}$$

where $[x_i x]\, r'$, $i = 1, \dots, n$, denotes the first divise difference of r'. Therefore, from (2.2) and theorem 3.2:

$$|r'(x)|_{I_i} \leq (C + 1)\, \omega(f'; h; I),\ i = 0, 1, \dots, n. \tag{3.10}$$

We are ready to prove the following convergence result.

Theorem 3.3. *Let $f \in C^1(I)$ and $s_f(x)$ the interpolating-derivative spline. There results*

$$\|f - s_f\|_\infty \leq (b - a)(C + 1)\, \omega(f'; h; I). \tag{3.11}$$

Proof. We can write, for $x \in I_i$, $i = 0, 1, ..., n$

$$|r(x)| = \left| \int_{x_0}^{x} r'(t)\, dt \right| \leq \max_{x \in I} |r'(x)|\, |x - x_{0 \leq}| \leq (b - a)(C + 1)\, \omega(f'; h; I) \tag{3.12}$$

and (3.11) is proved. □

We remark that the above theorems hold even when the partition Δ_n is quasi-uniform, i.e. such that: $\max\limits_{0 \leq i \leq n} \frac{h}{h_i}$ is bounded for $n \to \infty$ where $h_i = x_{i+1} - x_i$ and h is the norm of the partition. [E.Santi, M.G.Cimoroni: *Some new convergence results and applications of a class of interpolating-derivative splines.* In preparation].

We add now a property of the splines considered in this paper. The derivative-interpolating spline $s_f(x)$ defined in (2.3), considering a uniform partition Δ_n, reproduces any $f \in I\!P_2$. In fact, for $f = 1, x, x^2$ it is straightforward to verify that, $\underline{b}^*(a_0, a_n) = M = \underline{0}$. Therefore the coefficients of $s_f(x)|_{I_i}$ are:

$$\begin{cases} a_i = 0,\ b_i = 0, & c_i = 1, & \text{if} \quad f(x) = 1, \\ a_i = 0,\ b_i = 1, & c_i = x_i, & \text{if} \quad f(x) = x, \\ a_i = 2,\ b_i = 2x_i, & c_i = x_i^2, & \text{if} \quad f(x) = x^2, \end{cases}$$

and thus, for $x \in I_i$, $i = 0, \dots, n$:

$$\begin{cases} s_f(x) = 1, & \text{if} \quad f(x) = 1, \\ s_f(x) = (x - x_i) + x_i = x, & \text{if} \quad f(x) = x, \\ s_f(x) = \frac{2(x - x_i)^2}{2} + 2x_i(x - x_i) + x_i^2 = x^2, & \text{if} \quad f(x) = x^2. \end{cases}$$

4. **Numerical results**

We present now, some numerical results obtained by approximating some test functions by the spline considered in this paper. We denote $|r_n(x)| = |f(x) - s_f(x)|$ the error at x obtained by using a uniform partition of the interval $[-1, 1]$ in $n + 1$ subintervals. In table 1 we report the results relative to a test function $f(x)$ having only $f'(x) \in C[-1, 1]$ and considering different uniform partition with

$n = 4, 19, 39, 99$. In table 2 we report, for confirming the polynomial reproducibility, the results relative to a function $f(x) \in I\!P_2$ considering a uniform partition with $n = 4$. Table 3 contains the results relative to a more regular function $f(x) \in C^{\infty}[-1, 1]$ with different uniform partition, taking $n = 4, 19, 39, 79$.

Table 1

$f(x) = sign(x)x^2/2 + e^x$				
x	$\lvert \mathbf{r}_4(x) \rvert$	$\lvert \mathbf{r}_{19}(x) \rvert$	$\lvert \mathbf{r}_{39}(x) \rvert$	$\lvert \mathbf{r}_{99}(x) \rvert$
-1	0.0	0.0	0.0	0.0
-0.6	5.1 (-3)	1.2 (-4)	1.4 (-5)	8.6 (-7)
-0.2	1.6 (-3)	1.8 (-4)	1.5 (-5)	8.6 (-7)
0.2	2.1 (-2)	1.7 (-3)	4.3 (-4)	6.8 (-5)
0.6	1.7 (-2)	1.8 (-3)	4.3 (-4)	6.8 (-5)
1	5.6 (-3)	2.5 (-3)	5.2 (-4)	7.4 (-5)

Table 2

$f(x) = x^2 + 2x - 5$	
x	$\lvert \mathbf{r}_4(x) \rvert$
-1	0.0
-0.6	0.0
-0.2	8.9 (-16)
0.2	1.8 (-15)
0.6	8.9 (-16)
1	8.9 (-16)

Table 3

$f(x) = 1/(x^2 + 25)$				
x	$\lvert \mathbf{r}_4(x) \rvert$	$\lvert \mathbf{r}_{19}(x) \rvert$	$\lvert \mathbf{r}_{39}(x) \rvert$	$\lvert \mathbf{r}_{79}(x) \rvert$
-1	0.0	0.0	0.0	0.0
-0.6	1.8 (-5)	3.4 (-7)	4.4 (-8)	5.6 (-9)
-0.2	1.7 (-5)	3.4 (-7)	4.4 (-8)	5.6 (-9)
0.2	1.7 (-5)	3.4 (-7)	4.4 (-8)	5.6 (-9)
0.6	1.8 (-5)	3.4 (-7)	4.4 (-8)	5.6 (-9)
1	0.0	0.0	3.5 (-17)	0.0

5. Appendix

Proposition 2.2. *The infinitive norm of* $\widetilde{A}^{-1}$ *satisfies the following relation*

$$\frac{1}{6} \leq \left\| \widetilde{A}^{-1} \right\|_{\infty} \leq \frac{1}{2}. \tag{5.1}$$

Proof. It is straightforward to verify that $\left\| \widetilde{A} \right\|_{\infty} = 6$ and then, being $\left\| \widetilde{A}^{-1} \right\|_{\infty} \left\| \widetilde{A} \right\|_{\infty} \geq 1$, we obtain the left inequality in (5.1). For proving that $\left\| \widetilde{A}^{-1} \right\|_{\infty} \leq \frac{1}{2}$, we write $\widetilde{A} = 4I + H$, where

$$H = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \end{bmatrix}.$$

Thus, for all $\underline{x} : \|\underline{x}\|_\infty = 1$, there results

$$\left\|\widetilde{A}\underline{x}\right\|_\infty = \|(4I + H)\underline{x}\|_\infty \geq \|4\underline{x}\|_\infty - \|H\underline{x}\|_\infty \geq 2,$$

and then [7], (5.1) is proved. □

Proposition 2.3. *The entries* a_{1j}^{-1}, $j = 1, ..., n$ *of* $\widetilde{A}^{-1}$*have decreasing absolute values, the sign of* $(-1)^{j-1}$, *in particular, the following inequalities:*

$$\frac{1}{5} \leq a_{11}^{-1} = \underline{e}_1^T \widetilde{A}^{-1} \underline{e}_1 \leq \frac{1}{4}, \tag{5.2}$$

$$\left|a_{1n}^{-1}\right| = \left|\underline{e}_1^T \widetilde{A}^{-1} \underline{e}_n\right| \leq \begin{cases} \frac{1}{24} & \text{if} \quad n = 2, \\ \\ \frac{1}{90} & \text{if} \quad n \geq 3 \end{cases} \tag{5.3}$$

hold.

Proof. Using the results in [1], for the evaluation of the inverse matrix of a tridiagonal symmetric matrix, we can write

$$\widetilde{A}^{-1} = L + \underline{u}\underline{v}^T \tag{5.4}$$

and then,

$$a_{ij}^{-1} = l_{ij} + u_i v_j \tag{5.5}$$

where $l_{ij} = 0$ for $i \leq j$.

In our case, there results

$$u_1 = 1, \quad u_2 = -5, \quad u_i = -4u_{i-1} - u_{i-2}, \quad i = 3, ..., n \tag{5.6}$$

$$v_i = \alpha^{-1} u_{n-i+1}, \quad i = 1, ..., n \tag{5.7}$$

with $\alpha = 5u_n + u_{n-1}$.The matrix $\widetilde{A}^{-1}$ is symmetric and $a_{1i}^{-1} = a_{i1}^{-1} = \alpha^{-1} u_{n-i+1}$, $a_{in}^{-1} = a_{ni}^{-1} = \alpha^{-1} u_i$, $i = 1, 2, ..., n$.

Therefore, using proposition 2.1 , we deduce that $a_{11}^{-1} = a_{nn}^{-1} = u_n/\alpha$, $a_{1j}^{-1} = a_{n,n-j+1}^{-1}$, $j = 1, 2, ..., n$ are decreasing in absolute value and have the sign of $(-1)^{j-1}$, and, in particular, $a_{1n}^{-1} = 1/\alpha$.

For proving (5.2) consider that $a_{11}^{-1} = u_1 v_1 = u_n/\alpha = \frac{1}{5} \frac{5u_n + u_{n-1} - u_{n-1}}{5u_n + u_{n-1}} = \frac{1}{5}\left(1 - \frac{u_{n-1}}{5u_n + u_{n-1}}\right)$, and then, using (5.6), $0 < -\frac{u_{n-1}}{5u_n + u_{n-1}} = \frac{1}{4} \frac{u_n + u_{n-2}}{5u_n + u_{n-1}} < \frac{1}{4}$, (5.3) follows.

We get the inequality (5.3) considering that $\left|a_{1n}^{-1}\right| = \left|\frac{1}{5u_n + u_{n-1}}\right|$ and then, from (5.6), $\left|a_{1n}^{-1}\right| = \frac{1}{24}$ if $n = 2$, $\left|a_{1n}^{-1}\right| = \frac{1}{90}$ for $n = 3$, and $\left|a_{1n}^{-1}\right|$ decreases when n increases. Therefore the proposition is completely proved. □

Proposition 2.4. *Let* $C = \widetilde{A}^{-1} A^* \widetilde{A}^{-1}$. *For the entries* $c_{11} = \underline{e}_1^T C \underline{e}_1$, $c_{1n} = \underline{e}_1^T C \underline{e}_n$ *we have:*

$$0 < c_{11} < 1 \tag{5.8}$$

$$|c_{1n}| < c_{11}. \tag{5.9}$$

Proof. For $n \geq 3$, by (2.22), using (5.2) and (5.3), it is straightforward to verify that $0 < c_{11} = a_{11}^{-1} - \left[\left(a_{11}^{-1}\right)^2 + \left(a_{1n}^{-1}\right)^2 \right] < 1$ and $|c_{1n}| = \left| a_{1n}^{-1} \left(1 - 2a_{11}^{-1}\right) \right| < c_{11}$. □

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