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A CHARACTERIZATION OF π-CLOSED SCHUNCK CLASSES

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. A characterization of π-closed Schunck classes, followed by some consequences and applications in the formation theory of finite π-solvable groups are given.

1. Preliminaries

All groups considered in the paper are finite. Let π be a set of primes, π' the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G.

We first give some useful definitions.

Definition 1.1. ([9], [10], [12]) a) A class $\mathcal{X}$ of groups is a *homomorph* if $\mathcal{X}$ is epimorphically closed, i.e. if $G \in \mathcal{X}$ and N is a normal subgroup of G, then $G/N \in \mathcal{X}$.

b) A homomorph $\mathcal{X}$ is a *formation* if $G/N_1 \in \mathcal{X}$ and $G/N_2 \in \mathcal{X}$ imply $G/(N_1 \cap N_2) \in \mathcal{X}$.

c) A formation $\mathcal{X}$ is *saturated* if $\mathcal{X}$ is Frattini closed, i.e. if $G/\phi(G) \in \mathcal{X}$ implies $G \in \mathcal{X}$, where $\phi(G)$ denotes the Frattini subgroup of G.

d) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H with $core_G H = \{1\}$, where $core_G H = \cap\{H^g / g \in G\}$.

e) A homomorph $\mathcal{X}$ is a *Schunck class* if $\mathcal{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\mathcal{X}$, is itself in $\mathcal{X}$.

Definition 1.2. a) ([8]) A group G is π*-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of solvable group.

b) A class $\mathcal{X}$ of groups is said to be π*-closed* if

$$G/O_{\pi'} \in \mathcal{X} \;\Rightarrow\; G \in \mathcal{X}.$$

A π-closed homomorph, formation, respectively Schunck class is called π*-homomorph*, π*-formation*, respectively π*-Schunck class.*

Definition 1.3. ([9], [10]) Let $\mathcal{X}$ be a class of groups, G a group and H a subgroup of G.

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a) H is *$\mathcal{X}$-maximal* in G if: (i) $H \in \mathcal{X}$; (ii) $H \leq H^* \leq G$, $H^* \in \mathcal{X}$ imply $H = H^*$.

b) H is an *$\mathcal{X}$-projector* of G if, for any normal subgroup N of G, HN/N is $\mathcal{X}$-maximal in G/N.

c) H is an *$\mathcal{X}$-covering subgroup* of G if: (i) $H \in \mathcal{X}$; (ii) $H \leq K \leq G$, $K_0 \lhd K$, $K/K_0 \in \mathcal{X}$ imply $K = HK_0$.

The following results will be used in the paper.

Theorem 1.4. ([4]) *Let $\mathcal{X}$ be a class of groups, G a group and H a subgroup of G.*

a) If H is an $\mathcal{X}$-covering subgroup or an $\mathcal{X}$-projector of G, then H is $\mathcal{X}$-maximal in G.

b) If $\mathcal{X}$ is a homomorph, any $\mathcal{X}$-covering subgroup of G is an $\mathcal{X}$-projector of G.

Theorem 1.5. ([9]) *If $\mathcal{X}$ is a homomorph, G a group, N a normal subgroup of G, K/N an $\mathcal{X}$-covering subgroup of G/N and H is an $\mathcal{X}$-covering subgroup of K, then H is an $\mathcal{X}$-covering subgroup of G.*

Theorem 1.6. ([1]) *A solvable minimal normal subgroup of a group is abelian.*

Theorem 1.7. ([1]) *If S is a maximal subgroup of G with $\mathrm{core}_G S = \{1\}$ and N is a minimal normal subgroup of G, then $G = SN$ and $S \cap N = \{1\}$.*

Theorem 1.8. ([10]) *Let $\mathcal{X}$ be a class of groups. $\mathcal{X}$ is a saturated formation if and only if $\mathcal{X}$ is both a Schunck class and a formation.*

Theorem 1.9. ([2], [3], [4]) *Let $\mathcal{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\mathcal{X}$ is a Schunck class;

(2) any π-solvable group has $\mathcal{X}$-covering subgroups;

(3) any π-solvable group has $\mathcal{X}$-projectors.

2. The main result

In preparation for the main theorem of the paper, we give the following lemma.

Lemma 2.1. *Let $\mathcal{X}$ be a π-Schunck class, G a π-solvable group, such that $G \notin \mathcal{X}$, N a minimal normal subgroup of G with $G/N \in \mathcal{X}$ and H and $\mathcal{X}$-covering subgroup of G. Then H is a complement of N in G, i.e. $G = HN$ is $H \cap N = \{1\}$.*

Proof. Using that H is an $\mathcal{X}$-covering subgroup of G, from $H \leq G \leq G$, $N \lhd G$, $G/N \in \mathcal{X}$ follows that $G = HN$.

We prove now that $H \cap N = \{1\}$.

G is π-solvable group, hence the minimal normal subgroup N of G, being a chief factor of G, is either a solvable π-group or a π'-group. If we suppose that N is a π'-group, we obtain that $N \leq O_{\pi'}(G)$, hence

$$G/O_{\pi'} \cong (G/N)/(O_{\pi'}(G)/N).$$

But $G/N \in \mathcal{X}$ and $\mathcal{X}$ is a homomorph. So $G/O_{\pi'}(G) \in \mathcal{X}$, hence, $\mathcal{X}$ being π-closed, $G \in \mathcal{X}$, in contradiction with the hypothesis $G \notin \mathcal{X}$. It follows that N is a solvable π-group. By 1.6., N is abelian.

We prove that $H \cap N$ is a normal subgroup of G. Indeed, if $g \in G$ and $x \in H \cap N$, we have $g = nh$, with $n \in N$, $h \in H$ and

$$g^{-1}xg = (nh)^{-1}x(nh) = h^{-1}n^{-1}(xn)h = h^{-1}n^{-1}(nx)h = h^{-1}xh \in H \cap N,$$

where we used that N is abelian and that $H \cap N$ is normal in H.

Finally, N being a minimal normal subgroup of G and $H \cap N \triangleleft G$, $H \cap N \subseteq N$, we have $H \cap N = \{1\}$ or $H \cap N = N$. If we suppose that $H \cap N = N$, it follows that $N \subseteq H$, hence $G = HN = H$, a contradiction with $G \notin \mathcal{X}$ and $H \in \mathcal{X}$. So $H \cap N = \{1\}$. □

Theorem 2.2. *Let $\mathcal{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\mathcal{X}$ is a Schunck class;

(2) if G is a π-solvable group, $G \notin \mathcal{X}$ and N is a minimal normal subgroup of G such that $G/N \in \mathcal{X}$, then N has a complement in G;

(3) any π-solvable group G has $\mathcal{X}$-covering subgroups;

(4) any π-solvable group G has $\mathcal{X}$-projectors.

Proof. (1) implies (2). Let G be a π-solvable group, $G \notin \mathcal{X}$ and N a minimal normal subgroup of G such that $G/N \in \mathcal{X}$. By (1) and 1.9., G has an $\mathcal{X}$-covering subgroup H. By Lemma 2.1., H is a complement of N in G.

(2) implies (3). We prove by induction on $|G|$ that any π-solvable group G has $\mathcal{X}$-covering subgroups.

Two cases are possible:

1. $G \in \mathcal{X}$. In this case, G is its own $\mathcal{X}$-covering subgroup.

2. $G \notin \mathcal{X}$. Let N be a minimal normal subgroup of G. By the induction, G/N has an $\mathcal{X}$-covering subgroup E/N. We consider two possibilities:

a) $G/N \in \mathcal{X}$. Then, by 1.4.a) and 1.3.a), $E/N = G/N$. Applying (2) for the π-solvable group G, $G \notin \mathcal{X}$ and for its minimal normal subgroup N with $G/N \in \mathcal{X}$, we obtain that N has a complement V in G, i.e. $G = NV$ and $N \cap V = \{1\}$.

We notice that $V \in \mathcal{X}$, because

$$V \cong V/(N \cap V) \cong NV/N = G/N \in \mathcal{X}.$$

By 1.2.a), N is either a solvable π-group or a π'-group. If we suppose that N is a π'-group, then $N \leq O_{\pi'}(G)$ and so

$$G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N) \in \mathcal{X},$$

where we used that $G/N \in \mathcal{X}$ is a homomorph. Applying that $\mathcal{X}$ is π-closed, we get $G \in \mathcal{X}$, a contradiction. It follows that N is a solvable π-group, hence, by 1.6., N is abelian.

Let us consider two cases:

i) $core_G V \neq \{1\}$. By the induction, $G/core_G V$ has an $\mathcal{X}$-covering subgroup $H/core_G V$.

We notice that $H \neq G$, else $H = G$ implies $G/core_G V = H/core_G V \in \mathcal{X}$ and so $G/core_G V$ is its own $\mathcal{X}$-covering subgroup, hence, by 1.4.a), $G/core_G V$ is its own $\mathcal{X}$-maximal subgroup. But $V \in \mathcal{X}$ and $\mathcal{X}$ homomorph imply that $V/core_G V \in \mathcal{X}$. It follows that $V/core_G V = G/core_G V$ and so $V = G$, contradicting that $G \notin \mathcal{X}$ and $V \in \mathcal{X}$. Hence $H \neq G$.

The induction for H leads to the existence of an $\mathcal{X}$-covering subgroup L of H. Then $H/core_G V$ is an $\mathcal{X}$-covering subgroup of $G/core_G V$ and L is an $\mathcal{X}$-covering subgroup of H. Applying 1.5., we conclude that L is an $\mathcal{X}$-covering subgroup of G.

ii) $core_G V = \{1\}$. In this case, we prove that V is an $\mathcal{X}$-covering subgroup of G.

We proved that $V \in \mathcal{X}$.

Let now $V \leq K \leq G$, $K_0 \lhd K$ and $K/K_0 \in \mathcal{X}$. We shall prove that $K = VK_0$.

First, V is a maximal subgroup of G. Indeed, $V \neq G$, because $V \in \mathcal{X}$ and $G \notin \mathcal{X}$. Let now $V \leq V^* < G$. We show that $V = V^*$. Suppose $V < V^*$ and let $v^* \in V^* \setminus V \subset G = VN$ and put $v^* = vn$, where $v \in V$, $n \in N$. We have $n = v^{-1}v^* \in N \cap V^*$.

Let us prove that $N \cap V^* = \{1\}$. We notice that $G = NV \leq NV^* \leq G$ imply $G = NV^*$. Further, $N \cap V^*$ is a normal subgroup of G, because if $g \in G$, $x \in N \cap V^*$ we can prove that $g^{-1}xg \in N \cap V^*$. Indeed, if we take $g \in G = NV^*$ written as $g = mv^*$, with $m \in N$, $v^* \in V^*$, we have

$$g^{-1}xg = (mv^*)^{-1}(mv^*) = (v^*)^{-1}(m^{-1}x)mv^* =$$

$$= (v^*)^{-1}(xm^{-1})mv^* = (v^*)^{-1}xv^* \in N \cap V^*,$$

where we used that N is abelian and that $N \cap V^* \lhd V^*$. Hence $N \cap V^*$ is normal in G. N is a minimal normal subgroup of G and $N \cap V^* \subseteq N$. It follows that $N \cap V^* = \{1\}$ or $N \cap V^* = N$. But $N \cap V^* = N$ implies $N \subseteq V^*$ and so $G = NV^* = V^*$, in contradiction with the choice of V^*. Hence $N \cap V^* = \{1\}$.

From $n = v^{-1}v^* \in N \cap V^* = \{1\}$, we deduce $n = 1$ and so $v^{-1}v^* = 1$, which means $v^* = v \in V$, in contradiction with the choice of v^*. It follows that $V = V^*$. This completes the proof that V is a maximal subgroup of G.

By the above, we have for K with $V \leq K \leq G$ two possibilities: $K = V$ or $K = G$.

If $K = V$, we have $K_0 \lhd K = V$ and so $K = KK_0 = VK_0$.

If $K = G$, we reason as follows. Let us notice that $K_0 \neq \{1\}$, else

$$G = K \cong K/K_0 \in \mathcal{X},$$

a contradiction with $G \notin \mathcal{X}$. Let M be a minimal normal subgroup of G such that $M \subseteq K_0$. So we are in hypotheses of 1.7.: V is a maximal subgroup of G with $core_G V = \{1\}$ and M is a minimal normal subgroup of G. It follows that $G = VM$ and so

$$K = G = VM \leq VK_0 \leq G,$$

hence $K = G = VK_0$.

b) $G/N \notin \mathcal{X}$. In this case, we have $E/N \neq G/N$, because $E/N \in \mathcal{X}$. So $E \neq G$. By the induction, E has an $\mathcal{X}$-covering subgroup F. But E/N is an

$\mathcal{X}$-covering subgroup in G/N. Theorem 1.5. leads to the conclusion that F is an $\mathcal{X}$-covering subgroup of G.

(3) implies (4). Follows immediately from 1.9.

(4) implies (1). Follows also from 1.9. □

3. **Consequences**

Theorem 2.2. has some consequences on π-closed formations. In [5], we gave:

Theorem 3.1. ([5]) *Let $\mathcal{X}$ be a π-formation. The following conditions are equivalent:*

(1) $\mathcal{X}$ is saturated;

(2) if G is a π-solvable group and $G \notin \mathcal{X}$, but for the minimal normal subgroup N of G we have $G/N \in \mathcal{X}$, then N has a complement in G;

(3) any π-solvable group G has $\mathcal{X}$-covering subgroups.

From 2.2., 3.1. and 1.8., we obtain:

Corollary 3.2. *If $\mathcal{X}$ is a π-formation satisfying condition (2) from 2.2., then:*

a) $\mathcal{X}$ is a Schunck class;

b) $\mathcal{X}$ is Frattini closed, hence $\mathcal{X}$ is a saturated formation;

c) any π-solvable group G has $\mathcal{X}$-covering subgroups;

d) any π-solvable group G has $\mathcal{X}$-projectors.

4. **Some applications**

Finally, we give some applications of the main theorem of this paper, concerning to:

1. the existence and conjugacy given in [7] of $\mathcal{X}$-maximal subgroups in finite π-solvable groups, where $\mathcal{X}$ is a π-Schunck class;

2. the π-Schunck classes with the P property, introduced in [6].

4.1. In [7] we proved the following result:

Theorem 4.1.1. ([7]) *Let $\mathcal{X}$ be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in \mathcal{X}$. Then:*

(1) there is a subgroup S of G with $S \in \mathcal{X}$ and $AS = G$;

(2) there is an $\mathcal{X}$-maximal subgroup S of G with $AS = G$;

(3) if S_1 and S_2 are $\mathcal{X}$-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

Applying 4.1.1. and 2.2., we can prove the following theorem:

Theorem 4.1.2. *If $\mathcal{X}$ is a π-Schunck class, G is a π-solvable group, $G \notin \mathcal{X}$ and N is a minimal normal subgroup of G such that $G/N \in \mathcal{X}$, then:*

a) N has a complement H in G;

b) N is a solvable π-group, hence N is abelian;

c) H is $\mathcal{X}$-maximal in G;

d) H is conjugate to any $\mathcal{X}$-maximal subgroup S of G with $NS = G$.

Proof. a) Applying theorem 2.2., we obtain that N has a complement H in G, i.e. $HN = G$ and $H \cap N = \{1\}$.

b) N being a minimal normal subgroup of the π-solvable group G, N is either a solvable π-group, hence by 1.6. N is abelian, or N is a π'-group. We shall prove that the case N is π'-group is not possible in our hypotheses. Indeed, if we suppose that N is a π'-group, we have $N \leq O_{\pi'}(G)$ and

$$G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N) \in \mathcal{X},$$

hence, by the π-closure of $\mathcal{X}$, $G \in \mathcal{X}$, a contradiction.

c) In order to prove that H is $\mathcal{X}$-maximal in G. let us first notice that $H \in \mathcal{X}$. Indeed, we have

$$H \cong H/\{1\} = H/(H \cap N) \cong HN/N = G/N \in \mathcal{X}.$$

Let now $H \leq H^* \leq G$ and $H^* \in \mathcal{X}$. We prove that $H = H^*$. Suppose that $H < H^*$. Then there is an element $h^* \in H^* \setminus H \subset G = HN$ and $h^* = hn$, with $h \in H$, $n \in N$. Then $n = h^{-1}h^* \in H^* \cap N = \{1\}$ and so $n = 1$ and $h^* = h \in H$, in contradiction with the choice of h^*. The fact that $H^* \cap N = \{1\}$ follows from $H^* \cap N \lhd G$ (since N is abelian and $H^* \cap N \lhd H^*$) and from the hypotheses that N is a minimal normal subgroup of G.

d) Since we are in the hypotheses of 4.1.1, there is an $\mathcal{X}$-maximal subgroup S of G with $NS = G$. Applying now 4.1.1.(3), we conclude that H is conjugate to S. $\square$

4.2. In [6], we introduced the P property on a class $\mathcal{X}$ of groups. We say that $\mathcal{X}$ has the *P property* if, for any π-solvable group G, we have:

N minimal normal subgroup of G, N π'-group $\Rightarrow$ $G/N \in \mathcal{X}$.

Using theorem 2.2., we can prove the following result:

Theorem 4.2.1. *If $\mathcal{X}$ is a π-Schunck class with the P property and G is a π-solvable group, $G \notin \mathcal{X}$, then any minimal subgroup N of G which is a π'-group has a complement in G.*

Proof. By the P property, we have $G/N \in \mathcal{X}$. But $\mathcal{X}$ being a π-Schunck class, theorem 2.2. shows that $\mathcal{X}$ satisfies condition (2). Applying (2) for the π-solvable group G with $G \notin \mathcal{X}$ and for the minimal subgroup N of G with $G/N \in \mathcal{X}$, we conclude that N has a complement in G. $\square$

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