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A NEW SUBCLASS OF CONVEX FUNCTIONS

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Abstract. In this paper we have studied a class of univalent functions defined in the unit disc $U = \{z \in C : |z| < 1\}$.

1. Introduction

Let **A** be the class of the analytic functions in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$, which satisfy the conditions $f(0) = 0$ and $f'(0) = 1$.
We denote by K the class of univalent functions for which we have: $K \subset A$ and for every function $f \in K$ the domain $f(U)$ is a convex set in the complex plane.
It is well known that

$$K = \{f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)} > 0\right) \; for\ all\ z \in U\}.$$

We introduce the notation

$$\mathcal{K}_\lambda = \left\{ f \in A : (\exists)\lambda \in U, \left|\lambda|z|^2 + \left(1 - |z|^2\right)\frac{zf''(z)}{f'(z)}\right| < 1, (\forall) z \in U \right\}.$$

The condition which defines the class $\mathcal{K}_\lambda$ is a univalence criterion,whose proof and generalisation can be found in [4],[5].

2. Preliminaries

Definition 1. *Let f and g be two analytic functions in U. The function f is subordinate to g if there exists an analytic function denoted by Φ with the properties: $|\Phi(z)| < 1$, $z \in U$, $\Phi(0) = 0$ and $f(z) = g(\Phi(z)), z \in U$. The fact that f is subordinate to g will be denoted by $f \prec g$.*

Observation 1. If f and g are two analytic functions in U, g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$ then f is subordinate to g.

To prove our main result we will need the following lemmas.

Lemma A. *If the function f is analytic in U and $z_0 \in U$, then $z_0 f'(z_0)$ is the outward normal to the boundary of the domain $f(U_{r_0})$, where $r_0 = |z_0|$ and $U_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.*

Lemma B. (Miller and Mocanu) [2] *Let q be analytic and univalent in U. $q(0) = a$ and let $p(z) = a + p_n z^n + \dots$ be analytic in U with $p(z) \neq a, n \geq 1$. If $p \not\prec q$ then there exists points $r_0 e^{i\Theta_0} = z_0 \in U$ and $\zeta_0 \in \partial U$ and $m \geq n$ for which*

(i) $p(U_{r_0}) \subset q(U)$

(ii) $p(z_0) = q(\zeta_0)$

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(iii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.
If $q(z) = \frac{a+\bar{a}z}{1-z}$ *with* $Re(a) > 0$ *then* $q(U) = \{w \in \mathbb{C} : Re\ w > 0\}$ *and Lemma B becomes:*

Lemma B'. *Let p be analytic in U,* $p(z) = a + p_n z^n + \dots, p \not\equiv a, Re\ a > 0, n \geq 1$.
If $Re\ p(z) \not> 0, z \in U$ *then there exists* $z_0 \in U,\ x, y \in \mathbb{R}$ *for which*
(i) $p(z_0) = ix$
(ii) $z_0 p'(z_0) = y \leq -\frac{1}{2}[x^2 + 1]$

3. Main result

We observe that if $f_\delta(z) = \dfrac{e^{(1+\delta)z} - 1}{1+\delta}$ then $1 + \dfrac{z f_\delta''(z)}{f_\delta'(z)} = 1 + (1+\delta)z$ and so f_δ is not a convex function in U if $\delta > 0$.

Theorem 1. *If* $\lambda \in U$ *then* $\mathcal{K}_\lambda \nsubseteq K$.

Proof. We will prove that for $\lambda \in U$ exists a $\delta > 0$ for which $f_\delta \in \mathcal{K}_\lambda$.
If $\lambda \in U$ then $|\lambda| = 1 - \varepsilon, \epsilon \in (0,1)$ and from the triangle inequality results that:

$$\left|\lambda|z|^2 + \left(1 - |z|^2\right) \frac{z f_\delta''(z)}{f_\delta'(z)}\right| \leq (1-\varepsilon)|z|^2 + \left(1 - |z|^2\right)(1+\delta)|z|\,,\ z \in U. \tag{1}$$

Let $r = |z|$ and $g(r) = (1-\varepsilon)r^2 + (1-r^2)(1+\delta)r$. After calculations we get that $g(r) \leq (1+\delta)r(\delta)$ where $r(\delta)$ is the positive root of the equation $g'(r) = 0$. .
To show that there exists $\delta \in (0, +\infty)$ for which

$$\left|\lambda|z|^2 + (1-|z|^2)\frac{z f_\delta''(z)}{f_\delta'(z)}\right| < 1 \quad \text{for all } z \in U \tag{2}$$

it is enough to show the existence of δ with the property $(1+\delta)r(\delta) < 1$. The last assertion holds because:

$$\lim_{\delta \to o}(1+\delta)r(\delta) = \frac{|\lambda| + \sqrt{|\lambda|^2 + 3}}{3} < 1. \tag{3}$$

This completes the proof of the theorem .

Theorem 2. $\mathcal{K}_{-1} \subset K$.

Proof. 1. We will use Lemma B' to prove our assertion.
If we put $\lambda = -1$ and $p(z) = 1 + \dfrac{z f''(z)}{f'(z)}$ the inequality

$$\left|\lambda|z|^2 + \left(1 - |z|^2\right)\frac{z f''(z)}{f'(z)}\right| < 1, z \in U$$

may be rewritten in the following form

$$\left|-1 + \left(1 - |z|^2\right) p(z)\right| < 1, z \in U. \tag{4}$$

If $Re\ p(z) \not> 0, z \in U$ then according to Lemma B' there are $z_0 \in U$ and $x, y \in \mathbb{R}$ so that
(i) $p(z_0) = ix$

(ii) $z_0 p'(z_0) = y \leq \frac{-1}{2}\left(x^2+1\right)$
and we get that $\left|-1+\left(1-|z_0|^2\right)p(z_0)\right| = \left|-1+\left(1-|z_0|^2\right)ix\right| \geq 1$ which inequality is in contradiction with (4).

Theorem 3. *Let γ be a positive real number. The integral operation I defined by the equality*

$$I(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \tag{5}$$

satisfies the relation $I(\mathcal{K}_{-1}) \subset \mathcal{K}_{-1}$.

Proof. Let $f \in \mathcal{K}_{-1}$. We must show that the inequality

$$\left|-|z|^2+\left(1-|z|^2\right)\frac{zf''(z)}{f'(z)}\right| < 1, z \in U$$

implies that

$$\left|-|z|^2+\left(1-|z|^2\right)\frac{zF''(z)}{F'(z)}\right| < 1, z \in U.$$

Let $q(z) = \dfrac{zF''(z)}{F'(z)}$. We define the following set

$$B = \left\{r \in [0,+\infty) : \left|-|z|^2+\left(1-|z|^2\right)q(z)\right| < 1, (\forall) z \in \overline{U}_r\right\}$$

where $\overline{U}_r = \{z \in \mathbb{C} : |z| \leq r\}$. The set B isn't empty because $0 \in B$. Let $r_0 = \sup B$. For a fixed z the equality $\left|-|z|^2+\left(1-|z|^2\right)(x+iy)\right| = 1$ defines a circle in the $x0y$ system of coordinates. Let's denote this circle by $\mathcal{C}_z$. Because for all $z \in U$ the center $O_1\left(\frac{|z|^2}{1-|z|^2}, 0\right)$ of the circle $\mathcal{C}_z$ is on the real axis $0x$ and the point p(-1,0) is on the circle $\mathcal{C}_z$, we conclude that if $|z_1| < |z_2|$, then every point of the circle $\mathcal{C}_{z_1}$ except p are inside the circle $\mathcal{C}_{z_2}$. The above assertion shows that if $r_0 < 1$, then there exists $z_0 \in U$, $|z_0| = r_0$ so that $\left|-|z_0|^2+\left(1-|z_0|^2\right)q(z_0)\right| = 1$ and the domain $q(U_{r_0})$ is inside the circle $\mathcal{C}_{z_0}$. The border of the domain $q(U_r)$ is tangent to the circle $\mathcal{C}_{z_0}$ in the point $q(z_0)$ which implies that the outward normal $z_0q'(z_0)$ to the border of $q(U_{r_0})$ is outward normal to the circle $\mathcal{C}_{z_0}$. From (5) we get that :

$$q(z) + \frac{zq'(z)}{1+\gamma+q(z)} = \frac{zf''(z)}{f'(z)}, z \in U \tag{6}$$

We will prove that $Re\ \dfrac{1}{1+\gamma+q(z)} > 0, z \in U$.
If $Re\,(1+\gamma+q(z)) \not> 0$ for all $z \in U$ then we can apply Lemma B' and we get that there are $z_0 \in U$ and $x, y \in \mathbb{R}$ with the properties

(a) $Re\,(1+\gamma+q(z_0)) = ix$
(b) $z_0q'(z_0) = y \leq -\frac{1}{2}(x^2+1)$.

Replacing in (6) results that $Re\left(1+\gamma+q(z_0)+\dfrac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right) = Re\left(ix+\dfrac{y}{ix}\right) = 0$ on the other hand from (6) we get that:

$$Re\left(1+\gamma+q(z_0)+\frac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right) = Re\left(1+\gamma+\frac{z_0f''(z_0)}{f'(z_0)}\right) > 0$$

The contradiction shows that $Re\,(1+\gamma+q(z))>0$ for all $z\in U$. Let's return now to the proof of the theorem. The inequality $Re\,\dfrac{1}{1+\gamma+q(z_0)}>0$ is equivalent to :

$$\left|arg\,\frac{1}{1+\gamma+q(z_0)}\right|<\frac{\pi}{2} \tag{7}$$

Using (7) and the fact that $z_0q'(z_0)$ is the outward normal to the circle $\mathcal{C}_{z_0}$, we obtain that $q(z_0)+\dfrac{z_0q'(z_0)}{1+\gamma+q(z_0)}\notin Int\,\mathcal{C}_{z_0}$ or equivalently

$$\left|-|z_0|^2+\left(1-|z_0|^2\right)\left(q(z_0)+\frac{z_0q'(z_0)}{1+\gamma+q(z_0)}\right)\right|\geq 1$$

which implies that $\left|-|z_0|^2+\left(1-|z_0|^2\right)\dfrac{z_0f''(z_0)}{f'(z_0)}\right|\geq 1, z\in U$ in contradiction with the condition $f\in\mathcal{K}_{-1}$.

Conjecture. If $|\lambda|\leq 1$ and $\mathcal{K}_\lambda\subset K$ then $\lambda=-1$.

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