STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVIII**, Number 2, June 2003

ON THE DIRECT PRODUCT OF MULTIALGEBRAS

COSMIN PELEA

Abstract. This paper presents some properties of the direct product of a family of multialgebras of the same type.

1. **Introduction**

The multialgebras can be seen as relational systems which generalize the universal algebras. In the same way as in [4] the Cartesian product of a family of structures is organized as a structure it is possible to organize the Cartesian product of the supporting sets of a family of multialgebras as a multialgebra (see [9]). An important tool in the hyperstructure theory is the fundamental relation of a multialgebra (see [5]). The definition of this relation involves the term functions of the universal algebra of the nonempty sets of the given multialgebra and their images for some one element sets. These images of term functions are also used to obtain some identities that furnishes important classes of multialgebras. We will characterize them when our multialgebra is the direct product of a given family of multialgebras and we will prove that such an identity holds for the direct product if it holds for each member of the product.

We will also see that the definition of the multioperations in the direct product is natural in the way that the resulting multialgebra is the product in a category of multialgebras.

2. **Preliminaries**

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence with $n_\gamma \in \mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let A be a nonempty set and $P^*(A)$ the family of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set A of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations $f_\gamma,\ \gamma < o(\tau)$.

Of course, any universal algebra is a multialgebra (we can identify an one element set with its element).

As in [9] we can see the multialgebra $\mathfrak{A}$ as a relational system $(A, (r_\gamma)_{\gamma<o(\tau)})$ if we consider that, for any $\gamma < o(\tau)$, r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \ldots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow\ a_{n_\gamma} \in f_\gamma(a_0, \ldots, a_{n_\gamma-1}). \tag{1}$$

Received by the editors: 21.02.2003.

Key words and phrases. multialgebra, direct product, term function, identity.

Defining for any $\gamma < o(\tau)$ and for any $A_0, \dots, A_{n_\gamma - 1} \in P^*(A)$

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup \{ f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid \ a_i \in A_i, \ \forall i \in \{0, \dots, n_\gamma - 1\}\},$$

we obtain a universal algebra on $P^*(A)$ (see [7]). We denote this algebra by $\mathfrak{P}^*(A)$. As in [4], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$.

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{2}$$

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it results from [7], the multialgebra isomorphisms can be characterized as being those bijective homomorphisms for which (2) holds with equality.

Proposition 1. *For a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \dots, a_{n-1} \in A$ then*

$$h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1})).$$

Proof. We will use the steps of construction of a term.
Step 1. If $\mathbf{p} = \mathbf{x}_i$ $(i \in \{0, \dots, n-1\})$ then

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &= h(e_i^n(a_0, \dots, a_{n-1})) = h(a_i) \\ &= e_i^n(h(a_0), \dots, h(a_{n-1})) \\ &= p(h(a_0), \dots, h(a_{n-1})). \end{aligned}$$

Step 2. Suppose that the statement has been proved for $\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1} \in \mathbf{P}^{(n)}(\tau)$ and that $\mathbf{p} = f_\gamma(\mathbf{p}_0, \dots, \mathbf{p}_{n_\gamma-1})$. Then we have

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &= h(f_\gamma(p_0, \dots, p_{n_\gamma-1})(a_0, \dots, a_{n-1})) \\ &= h(f_\gamma(p_0(a_0, \dots, a_{n-1}), \dots, p_{n_\gamma-1}(a_0, \dots, a_{n-1}))) \\ &= h(\bigcup \{ f_\gamma(b_0, \dots, b_{n_\gamma-1}) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\}) \\ &= \bigcup \{ h(f_\gamma(b_0, \dots, b_{n_\gamma-1})) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\} \\ &\subseteq \bigcup \{ f_\gamma(h(b_0), \dots, h(b_{n_\gamma-1}))) \mid b_i \in p_i(a_0, \dots, a_{n-1}), \ i \in \{0, \dots, n_\gamma - 1\}\}. \end{aligned}$$

Since for any $i \in \{0, \dots, n_\gamma - 1\}$, $b_i \in p_i(a_0, \dots, a_{n-1})$ it follows

$$h(b_i) \in h(p_i(a_0, \dots, a_{n-1})) \subseteq p_i(h(a_0), \dots, h(a_{n-1})));$$

so we have,

$$\begin{aligned} h(p(a_0, \dots, a_{n-1})) &\subseteq f_\gamma(p_0(h(a_0), \dots, h(a_{n-1})), \dots, p_{n_\gamma-1}(h(a_0), \dots, h(a_{n-1}))) \\ &= f_\gamma(p_0, \dots, p_{n_\gamma-1})(h(a_0), \dots, h(a_{n-1})) \\ &= p(h(a_0), \dots, h(a_{n-1})) \end{aligned}$$

which finishes the proof. $\square$

Remark 1. If for any $\gamma < o(\tau)$ and for all $a_0, \ldots, a_{n_\gamma - 1} \in A$ we have equality in (2), then

$$h(p(a_0, \ldots, a_{n-1})) = p(h(a_0), \ldots, h(a_{n-1})).$$

The proof can be done as before, but it it also results from some properties that can be established for the universal algebra $\mathfrak{P}^*(A)$ (see [1]).

We can easily construct the category of the multialgebras of the same type τ where the morphisms are considered to be the homomorphisms and the composition of two morphisms is the usual mapping composition and we will denote it by $\mathbf{Malg}(\tau)$.

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. Using the model offered by [4] and looking at the definitions of the hyperstructures from [2] and also at the generalizations presented in [10], named H_v-structures, we can consider that the n-ary (strong) identity

$$\mathbf{q} = \mathbf{r}$$

is said to be satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1})$$

for all $a_0, \ldots, a_{n-1} \in A$, where q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$. We can also consider that a weak identity (the notation is intended to be as suggestive as possible)

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is said to be satisfied on a multialgebra $\mathfrak{A}$ if

$$q(a_0, \ldots, a_{n-1}) \cap r(a_0, \ldots, a_{n-1}) \neq \emptyset$$

for all $a_0, \ldots, a_{n-1} \in A$, where q and r have the same signification as before. Many important particular multialgebras are defined as being those multialgebras which satisfy a given set of identities.

3. Direct products of multialgebras

Given a family of relational systems of the same type $\tau = (n_\gamma + 1)_{\gamma < o(\tau)}$, $(\mathfrak{A}_i = (A_i, (r_\gamma)_{\gamma < o(\tau)}) \mid i \in I)$, in [4] is defined the direct product of this family as being the relational system obtained on the Cartesian product $\prod_{i \in I} A_i$ considering that for $(a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma})_{i \in I} \in \prod_{i \in I} A_i$,

$$((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma})_{i \in I}) \in r_\gamma \Leftrightarrow (a_i^0, \ldots, a_i^{n_\gamma}) \in r_\gamma, \ \forall i \in I.$$

If we consider a family $\{\mathfrak{A}_i\}_{i \in I}$ of multialgebras of type τ and the relational systems defined by (1), the relational system that results on the Cartesian product $\prod_{i \in I} A_i$ from the above considerations is a multialgebra of type τ with the multioperations:

$$f_\gamma((a_i^0)_{i \in I}, \ldots, (a_i^{n_\gamma - 1})_{i \in I}) = \prod_{i \in I} f_\gamma(a_i^0, \ldots, a_i^{n_\gamma - 1}), \tag{3}$$

for any $\gamma < o(\tau)$. This multialgebra is called the direct product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. We observe that the canonical projections of the product, e_i^I, $i \in I$, are multialgebra homomorphisms.

Proposition 2. *The multialgebra $\prod_{i \in I} \mathfrak{A}_i$ constructed this way, together with the canonical projections, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category* $\mathbf{Malg}(\tau)$.

Proof. For any multialgebra $\mathfrak{B}$ and for any family of multialgebra homomorphisms $(\alpha_i : B \to A_i \mid i \in I)$ there is only one homomorphism $\alpha\colon B \to \prod_{i\in I} A_i$ such that $\alpha_i = e_i^I \circ \alpha$ for any $i \in I$.

Indeed, there exists only one mapping α such that the diagram

$$\begin{tikzcd} \prod_{i\in I} A_i \arrow[r, "e_i^I"] & A_i \\ B \arrow[u, "\alpha"] \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

is commutative. This mapping is defined by $\alpha(b) = (\alpha_i(b))_{i\in I}$. Now, all we have to do is to verify that α is a multialgebra homomorphism. If we consider $\gamma < o(\tau)$ and $b_0, \ldots, b_{n_\gamma-1} \in B$ then

$$\begin{aligned}\alpha(f_\gamma(b_0, \ldots, b_{n_\gamma-1})) &= \{\alpha(b) \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1})\}\\ &= \{(\alpha_i(b))_{i\in I} \mid b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1})\}.\end{aligned}$$

From $b \in f_\gamma(b_0, \ldots, b_{n_\gamma-1})$ it follows that for any $i \in I$,

$$\alpha_i(b) \in \alpha_i(f_\gamma(b_0, \ldots, b_{n_\gamma-1})) \subseteq f_\gamma(\alpha_i(b_0), \ldots, \alpha_i(b_{n_\gamma-1})),$$

so we have

$$\begin{aligned}\alpha(f_\gamma(b_0, \ldots, b_{n_\gamma-1})) &\subseteq \prod_{i\in I} f_\gamma(\alpha_i(b_0), \ldots, \alpha_i(b_{n_\gamma-1}))\\ &= f_\gamma((\alpha_i(b_0))_{i\in I}, \ldots, (\alpha_i(b_{n_\gamma-1}))_{i\in I})\\ &= f_\gamma(\alpha(b_0), \ldots, \alpha(b_{n_\gamma-1}))\end{aligned}$$

which finishes the proof. $\square$

Lemma 1. *For every $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$, we have*

$$p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1}). \tag{4}$$

Proof. We will use again the steps of construction of a term.

Step 1. If $\mathbf{p} = \mathbf{x}_j$ $(j \in \{0, \ldots, n-1\})$ then

$$\begin{aligned}&p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = e_j^n((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) = (a_i^j)_{i\in I}\\ &=(e_j^n(a_i^0, \ldots, a_i^{n-1}))_{i\in I} = \prod_{i\in I} e_j^n(a_i^0, \ldots, a_i^{n-1}) = \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1}).\end{aligned}$$

Step 2. Suppose that the statement has been proved for $\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1} \in \mathbf{P}^{(n)}(\tau)$ and that $\mathbf{p} = f_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma-1})$. Then we have

$$\begin{aligned}p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) &= f_\gamma(p_0, \ldots, p_{n_\gamma-1})((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I})\\ &= f_\gamma(p_0((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}), \ldots, p_{n_\gamma-1}((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}))\\ &= f_\gamma(\prod_{i\in I} p_0(a_i^0, \ldots, a_i^{n-1}), \ldots, \prod_{i\in I} p_{n_\gamma-1}(a_i^0, \ldots, a_i^{n-1}))\end{aligned}$$

But

$$(x_i)_{i\in I} \in f_\gamma(\prod_{i\in I} p_0(a_i^0, \ldots, a_i^{n-1}), \ldots, \prod_{i\in I} p_{n_\gamma-1}(a_i^0, \ldots, a_i^{n-1}))$$

if and only if for each $j \in \{0, \ldots, n_\gamma - 1\}$ and $i \in I$, there exists some $b_i^j \in p_j(a_i^0, \ldots, a_i^{n-1})$ such that

$$(x_i)_{i\in I} \in f_\gamma((b_i^0)_{i\in I}, \ldots, (b_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(b_i^0, \ldots, b_i^{n_\gamma-1}),$$

thus

$$\begin{aligned} p((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) &= \prod_{i\in I} f_\gamma(p_0(a_i^0, \ldots, a_i^{n-1}), \ldots, p_{n_\gamma-1}(a_i^0, \ldots, a_i^{n-1})) \\ &= \prod_{i\in I} f_\gamma(p_0, \ldots, p_{n_\gamma-1})(a_i^0, \ldots, a_i^{n-1}) \\ &= \prod_{i\in I} p(a_i^0, \ldots, a_i^{n-1}) \end{aligned}$$

which finishes the proof of the lemma. □

Proposition 3. *If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is also satisfied on the multialgebra $\prod_{i\in I} \mathfrak{A}_i$.*

Proof. Let us consider that $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$, where $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. This means that for all $i \in I$ and for any $a_i^0, \ldots, a_i^{n-1} \in A_i$ we have $q(a_i^0, \ldots, a_i^{n-1}) \cap r(a_i^0, \ldots, a_i^{n-1}) \neq \emptyset$. Using Lemma 1, it follows that

$$\begin{aligned} &q((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) \cap r((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) \\ &\quad = \prod_{i\in I} q(a_i^0, \ldots, a_i^{n-1}) \cap \prod_{i\in I} r(a_i^0, \ldots, a_i^{n-1}) \\ &\quad = \prod_{i\in I} (q(a_i^0, \ldots, a_i^{n-1}) \cap r(a_i^0, \ldots, a_i^{n-1})) \neq \emptyset \end{aligned}$$

and the statement is proved. □

Proposition 4. *If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} = \mathbf{r}$ is also satisfied on the multialgebra $\prod_{i\in I} \mathfrak{A}_i$.*

Proof. Consider that $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. For all $i \in I$ and for any $a_i^0, \ldots, a_i^{n-1} \in A_i$ we have $q(a_i^0, \ldots, a_i^{n-1}) = r(a_i^0, \ldots, a_i^{n-1})$. Using Lemma 1, it follows that

$$\begin{aligned} q((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) &= \prod_{i\in I} q(a_i^0, \ldots, a_i^{n-1}) = \prod_{i\in I} r(a_i^0, \ldots, a_i^{n-1}) \\ &= r((a_i^0)_{i\in I}, \ldots, (a_i^{n-1})_{i\in I}) \end{aligned}$$

and the statement is proved. □

References

[1] Breaz, S.; Pelea, C. Multialgebras and term functions over the algebra of their nonvoid subsets, *Mathematica (Cluj)*, to appear.

[2] Corsini, P. Prolegomena of hypergroup theory. Supplement to Riv. Mat. Pura Appl. *Aviani Editore, Tricesimo*, 1993.

[3] Grätzer, G. A representation theorem for multi-algebras. *Arch. Math.* **3**, 1962, 452–456.

[4] Grätzer, G. Universal algebra. *D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London* 1968.

[5] Pelea, C. On the fundamental relation of a multialgebra, *Ital. J. Pure Appl. Math.* **10** 2001, 141–146.

[6] Purdea, Ioan; Pic, Gheorghe Tratat de algebră modernă. Vol. I. (Romanian) [Treatise on modern algebra. Vol. I] *Editura Academiei Republicii Socialiste România, Bucharest*, 1977.

[7] Pickett, H. E. Homomorphisms and subalgebras of multialgebras. *Pacific J. Math.* **21** 1967, 327–342.

[8] Purdea, Ioan Tratat de algebră moderna. Vol. II. (Romanian) [Treatise on modern algebra. Vol. II] With an English summary. *Editura Academiei Republicii Socialiste România, Bucharest*, 1982.

[9] Schweigert, D. Congruence relations of multialgebras, *Discrete Math.*, **53** 1985, 249–253.

[10] Vougiouklis, T. Construction of H_v-structures with desired fundamental structures,*New frontiers in hyperstructures (Molise, 1995)*, 177–188, Ser. New Front. Adv. Math. Ist. Ric. Base,*Hadronic Press, Palm Harbor, FL*, 1996.

"Babeş–Bolyai" University, Faculty of Mathematics
and Computer Science, Str. Mihail Kogălniceanu nr. 1,
RO–3400 Cluj–Napoca, Romania
E-mail address: cpelea@math.ubbcluj.ro