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APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS

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Abstract. In this paper, by using the method of differential subordinations, we obtain a more general condition, from which it could be found the conditions for starlikenes [2], [3], [4].

1. Introduction and definitions

Let $\mathbf{U}$ denote the open unit disk.

Let $H = H(\mathbf{U})$ denote the class of functions analytic in $\mathbf{U}$

For n a positive integer and $a \in \mathbb{C}$, let

$$H\,[a,n] = \left\{\; f \in H \;;\; f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \right\}$$

with $H_0 = H\,[0,1]$

Recall the concept of subordination :

Let f and g be in H. The function f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function φ analytic in $\mathbf{U}$, with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, such that $f(z) = g(\varphi(z))$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbf{U}) \subset g(\mathbf{U})$.

Let $\Psi : \mathbb{C}^3 \times \mathbf{U} \to \mathbb{C}$ and let h be univalent in $\mathbf{U}$. If p is analytic in $\mathbf{U}$ and satisfies the (second-order) differential subordination

$$\Psi(p(z), zp'(z), z^2p"(z), z) \prec h(z) \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordinations, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1).(Note that the best dominant is unique up to a rotation of $\mathbf{U}$).

If we require the more restrictive condition $p \in H[a,n]$, then p will be called an (a,n) - solution, q an (a,n) - dominant and $\widetilde{q}$ the best (a,n) - dominant.

Theorem A. (M. Obradovic, T. Yaguchi, H. Saitoh [4]). Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$ and $\mathrm{Re}[q(z)] > \frac{1}{2}$, $z \in \mathbf{U}$.If $0 \leq \alpha < 1$, p is analytic in $\mathbf{U}$ with $p(0) = 1$ and if

$$(1-\alpha)p^2(z) + (2\alpha - 1)p(z) - \alpha + (1-\alpha)zp'(z) \prec$$
$$(1-\alpha)q^2(z) + (2\alpha - 1)q(z) - \alpha + (1-\alpha)zq'(z) \equiv h(z), \tag{2}$$

then $p \prec q$ and q is the best dominant of (2).

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Theorem B. (Sufficient conditions for starlikeness, P. T. Mocanu, Gh. Oros [2]). Let the function $h(z) = 1+(2\alpha+1)\mu z+\alpha\mu^2 z^2$ where $\alpha > 0$ and $0 < \mu \leq 1+\frac{1}{2\alpha}$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in $\mathbf{U}$ and satisfies the condition

$$\alpha zp'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec h(z),$$

then $p(z) \prec 1 + \mu z$ and this result is sharp.

Theorem C. (Sufficient conditions for starlikeness II, P. T. Mocanu, Gh. Oros [3]) Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$, $\mathrm{Re}\; q(z) > \frac{\alpha-\beta}{2\alpha}$; $\alpha > 0, \alpha+\beta > 0$ and let

$$h(z) = \alpha nzq'(z) + \alpha q^2(z) + (\beta-\alpha)q(z). \tag{3}$$

If the function $p(z) = 1 + p_n z^n + \ldots$ satisfies the condition :

$$\alpha zp'(z) + \alpha p^2(z) + (\beta-\alpha)p(z) \prec h(z),$$

where h is given by (3) ,then $p(z) \prec q(z)$ where q is the best dominant.

Theorem D. (S. S. Miller, P. T. Mocanu, [1]) Let q be univalent in $\mathbf{U}$ and let θ in Φ be analytic in a domain D containing $q(\mathbf{U})$, with $\Phi(w) \neq 0$, when $w \in q(\mathbf{U})$. Set $Q(z) = zq(z)\Phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that

i) Q is starlike in $\mathbf{U}$, and

ii) $\mathrm{Re}\,\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left[\frac{\theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] > 0, \quad z \in \mathbf{U}.$

If p is analytic in $\mathbf{U}$, with $p(0) = q(0)$, $p(\mathbf{U}) \subset D$ and

$$\theta[p(z)] + zp'(z)\Phi[p(z)] \prec \theta[q(z)] + zq'(z)\Phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

2. Main Results

Theorem 1. Let q be convex in $\mathbf{U}$, with $\mathrm{Re}[2aq(z) + b] > 0$, $q(0) = 1$, when $a, b \in \mathbb{C}$, $a \neq 0$ and let

$$h(z) = aq^2(z) + bq(z) + czq'(z); \quad c > 0.$$

If the function $p \in H[1, n]$, i.e. $p(z) = 1 + p_n z^n + \ldots$ satisfies the differential subordination :

$$ap^2(z) + bp(z) + czp'(z) \prec h(z),$$

then $p \prec q$ and q is the best $(1, n)$ - dominant.

Proof. (On checking the conditions of Th.D)

Let

$$\theta(w) = aw^2 + bw$$

$$\Phi(w) = c \neq 0 \;,\; \forall w \in q(\mathbf{U})$$

$$Q(z) = zq'(z)\Phi[q(z)] = czq'(z)$$

$i)$; $Q(z) = czq'(z)$ is starlike because $q(z)$ is convex and $c > 0$.

$$ii)\;\mathrm{Re}\,\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left[\frac{\theta'[q(z)]}{\Phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] = \mathrm{Re}\left[\frac{2aq(z)+b}{c} + z\frac{Q'(z)}{Q(z)}\right] =$$

$$= \mathrm{Re}\left[\frac{2aq(z)}{c} + \frac{b}{c} + z\frac{Q'(z)}{Q(z)}\right] > 0,$$

because

$$\mathrm{Re}[2aq(z) + b] > 0.$$

The conditions of Th.D is satisfied and we have: for $p \in H[1,n]$ which satisfies $ap^2(z) + bp(z) + czp'(z) \prec h(z)$ we have $p \prec q$ and q is the best $(1,n)$ - dominant.

Remark 1. For $q(z) = 1+\mu z$ univalent in $\mathbf{U}$, with $\alpha > 0$ and $0 < \mu \leq 1+1/\alpha$, if

$a = \alpha$
$b = 1 - \alpha$
$\Phi(w) = \alpha$

we obtain the result given in Theorem B (P. T. Mocanu, Gh. Oros, [2])

Remark 2. For q convex in $\mathbf{U}$, with $q(0) = 1$, Re $q(z) > \frac{\alpha-\beta}{2\alpha}$, if

$a = \alpha$
$b = \beta - \alpha$
$\Phi(w) = \alpha \cdot u$, when $\alpha > 0$ and $\alpha + \beta > 0$,

we reobtain the Theorem C (P. T. Mocanu, Gh. Oros, [3])

Remark 3. Let q be a convex function in $\mathbf{U}$, with $q(0) = 1$ and Re $q(z) > \frac{1}{2}$; $z \in \mathbf{U}$

If $a = 1 - \alpha$
$b = 2\alpha - 1 \quad ; \alpha \in [0, 1)$
$\theta(w) = (1-\alpha)w^2 + (2\alpha - 1)w - \alpha$
$\Phi(w) = 1 - \alpha$,

we obtain the condition for starlikeness of Theorem A. (M. Obradovic, T. Yaguchi, H. Saitoh [4]).

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