STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVIII**, Number 2, June 2003

ON THE NECESSARY AND SUFFICIENT CONDITION FOR THE REGULARITY OF A MULTIDIMENSIONAL INTERPOLATION SCHEME

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Abstract. In this article we will present the necessary and sufficient condition for the regularity of a multidimensional interpolation scheme, in the case when the interpolation indexes are taken from an arbitrary set S from $\mathbb{N}^d$. In particular, if the index set S (of the interpolation space $\mathcal{P}_S$) is inferior with respect to $\mathbb{N}^d$, we obtain the theorem 3.4.2. from [1]. The set Δ_k^d from $\mathbb{N}^d$, given by the relation (1), together with the proposition 1, are the key elements that allow us to approach the theorem in a general context and to give another proof for it, compared of course with the one given in [1].

Let $\mathbb{N}^d = \{\mathbf{i} = (i_1, i_2, \ldots, i_d)/i_k \geq 0, i_k \in \mathbb{N}, k = \overline{1,d}\}$, $|\mathbf{i}| = i_1 + i_2 + \ldots + i_d$, and the definitions:

1. *We say that* $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^d$ *is in the relation "$\leq$" with* $\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d$ *and we write* $\mathbf{i} \leq \mathbf{j}$ *when* $0 \leq i_k \leq j_k$ *for any* $k = \overline{1,d}$.

2. *We say that* $I \subset \mathbb{N}^d$ *is a lower set with respect to* $\mathbb{N}^d$ *if for any* $\mathbf{i} \in I$ *and* $\mathbf{j} \in \mathbb{N}^d$ *so that* $\mathbf{j} \leq \mathbf{i}$, *we have* $\mathbf{j} \in I$. *In the same manner we can define the inferior set with respect to any set* $S \subset \mathbb{N}^d$

If $\Delta_k^d \subset \mathbb{N}^d$,

$$\Delta_k^d = \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \ldots \bigcup_{i_{d-1}=0}^{k-(i_1+\ldots+i_{d-2})} \{i_1, \ldots, i_{d-1}, k - (i_1 + \ldots + i_{d-1})\} \tag{1}$$

we have $T_n^d = \bigcup_{k=0}^{n} \Delta_k^d$, that is

$$T_n^d = \bigcup_{k=0}^{n} \Delta_k^d = \bigcup_{k=0}^{n} \bigcup_{i_1=0}^{k} \bigcup_{i_2=0}^{k-i_1} \ldots \bigcup_{i_{d-1}=0}^{k-(i_1+\ldots+i_{d-2})} \{i_1, \ldots, i_{d-1}, k - (i_1 + \ldots + i_{d-1})\}$$

is a lower set with respect to $\mathbb{N}^d$.

Proposition 1. *Any set S from $\mathbb{N}^d$ can be written in the form*

$$S = \bigcup_{t=1}^{n} {\Delta'}_{k_t}^{d} \tag{2}$$

where $k_n = \max_{i \in S} |\mathbf{i}|$, $0 \leq k_1 \leq \ldots \leq k_n$, ${\Delta'}_{k_t}^d \subset \Delta_{k_t}^d$ *and* $\Delta_{k_t}^d$, $t = \overline{1,n}$, *are given by* (1).

Received by the editors: 07.01.2003.

Proposition 2. *If $P \in \mathcal{P}_S$ and $S \subset \mathbb{N}^d$ given by (2), then for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in S \subset \mathbb{N}^d$, we have*

$$P(\mathbf{x}) = \sum_{\mathbf{i}\in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d} = \sum_{t=1}^{n} P_{k_t}(\mathbf{x}),\ P_{k_t}(\mathbf{x}) = \sum_{\mathbf{i}\in {\Delta'}^d_{k_t}} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d},\ t = \overline{1,n}$$

The $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in S \subset \mathbb{N}^d$ order derivatives of P are:

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}} P(\mathbf{x}) =$$

$$= \sum_{\mathbf{i}\in S, \boldsymbol{\alpha}\le \mathbf{i}} a_{\mathbf{i}} \frac{i_1!}{(i_1-\alpha_1)!} \frac{i_2!}{(i_2-\alpha_2)!} \cdots \frac{i_d!}{(i_d-\alpha_d)!} x_1^{i_1-\alpha_1} x_2^{i_2-\alpha_2} \ldots x_d^{i_d-\alpha_d},$$

and those of those of P_{k_t} are:

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}} P_{k_t}(\mathbf{x}) = \frac{\partial^{k_t}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_{d-1}^{\alpha_{d-1}} \partial x_d^{k_t-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}} P_{k_t}(\mathbf{x}) =$$

$$= \alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_t - (\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\, a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_t-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})},$$

for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

In what follows, whenever we write S we will denote an arbitrary set from $\mathbb{N}^d$. By extending from a set I inferior with respect to $\mathbb{N}^d$ (*see also* [1]) to an arbitrary set S from $\mathbb{N}^d$, we define the following notions.

Definition 1. *A polinomial multidimensional interpolation scheme $(E, \mathcal{P}_S)$ consists of:*

(a) *A set of nodes $Z = \{\mathbf{x}_q\}_{q=1}^m = \{(x_{q,1}, x_{q,2}, \ldots, x_{q,d})\}_{q=1}^m$ from $\mathbb{R}^d$*

(b) *An interpolation space*

$$\mathcal{P}_S = \left\{ P/P(\mathbf{x}) = \sum_{\mathbf{i}\in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d},\ a_{\mathbf{i}} \in \mathbb{R},\ \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \right\},$$

which is the space of the d variables polinomes with real coefficients where S is an arbitrary subset of $\mathbb{N}^d$, and

(c) *An incidence matrix $E = (e_{q,\boldsymbol{\alpha}})$, $1 \le q \le m$, $\boldsymbol{\alpha} \in S$, where $e_{q,\boldsymbol{\alpha}} = 0$ or 1.*

The interpolation problem associated with $(E, \mathcal{P}_S)$ consists of finding the polinomes $P \in \mathcal{P}_S$, that would satisfy the equations

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}} P(\mathbf{x}_q) = \mathbf{c}_{q,\boldsymbol{\alpha}} \tag{3}$$

for any $q = \overline{1,m}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in S$ with $e_{q,\boldsymbol{\alpha}} = 1$ where $c_{q,\boldsymbol{\alpha}}$ are arbitrary real constants

The (3) equations make up a system of linear equations whose unknowns are the real coefficients of the polinome P. The matrix $\mathcal{M}(E,Z)$ of this system is at the same time the matrix of the interpolation scheme $(E, \mathcal{P}_S)$, and it is called *Vandermonde matrix.* If $\mathcal{M}(E,Z)$ is a square matrix, then its determinant, $det\mathcal{M}(E,Z) = \mathcal{D}(E,Z)$, is the determinant of the system (3), and of the interpolation scheme $(E, \mathcal{P}_S)$, and it is called *Vandermonde determinant.*

If in definition 1 we consider that S is a set I inferior with respect to $\mathbb{N}^d$, then, according to [1], $(E, \mathcal{P}_I)$ is a *Birkhoff interpolation scheme*, the E matrix is the *Birkhoff incidence matrix*, and the interpolation polynom P is a *Birkhoff polinome*. Also, according to [1], by various particularisations of the Birkhoff interpolation scheme, we obtain the *Lagrange, Hermite, Taylor* and *Abel interpolation schemes*, with their corresponding incidence matrices and interpolation polinomes.

Definition 2. *Let S be a set from $\mathbb{N}^d$ and $(E, \mathcal{P}_S)$ the corresponding multidimensional interpolation scheme. We say that $E = (e_{q,\boldsymbol{\alpha}})$ is an Abel incidence matrix, if for any $\boldsymbol{\alpha} \in S$, $e_{q,\boldsymbol{\alpha}} = 1$ for exactly one $q \in \overline{1,m}$. The scheme, the polinome and the interpolation problem corresponding to the Abel incidence matrix, are called Abel interpolation scheme, Abel interpolation polinome, respectively Abel interpolation problem.*

Definition 3. *The multidimensional interpolation scheme $(E, \mathcal{P}_S)$ is called normal if $|E| = \dim \mathcal{P}_S$.*

Because in the present article we will work only with normal interpolation schemes, from now on, whenever we discuss an interpolation scheme, we will consider it normal.

Definition 4. *We say that an interpolation scheme $(E, \mathcal{P}_S)$ is*

(a) *singular, if $\mathcal{D}(E, Z) = 0$ for any choice of the set of nodes Z,*
(b) *regular, if $\mathcal{D}(E, Z) \neq 0$ for any choice of the set of nodes Z and*
(c) *almost regular, if $\mathcal{D}(E, Z) \neq 0$ for almost all choices of the set of nodes Z.*

Definition 5. *Two interpolation schemes are equivalent when the systems of their interpolation problems are equivalent.*

Theorem 1. *The interpolation scheme $(E, \mathcal{P}_S)$ is regular if and only if it is equivalent with an Abel interpolation scheme.*

Proof. If E is a Abel matrix and if the order of the coefficients of the interpolation polinome and the order of the derivatives from the interpolation system are those which correspond to the order of the elements of the set S, then the matrix of the interpolation system is superior triangular. If not, by switching lines or (and) columns in $\mathcal{M}(E, Z)$, we can obtain what we have previously shown, and thus it follows that the new determinant is different from the previous one only through its sign. It follows that the determinant of this matrix is:

$$d_S = \pm \prod_{\boldsymbol{\alpha} \in S} \boldsymbol{\alpha}!$$

where $\boldsymbol{\alpha}! = \alpha_1! \alpha_2! \ldots \alpha_d!$. It follows that $d_S \neq 0$, and as a result $(E, \mathcal{P}_S)$ is regular.

Conversely, we assume that $(E, \mathcal{P}_S)$ is regular and let P be the solution of the interpolation problem of this interpolation scheme, where:

$$P(\mathbf{x}) = P(x_1, x_2, \ldots, x_d) = \sum_{\mathbf{i} \in S} a_{\mathbf{i}} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d}$$

for any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $a_{\mathbf{i}} \in \mathbb{R}^d$ (real constants, since the given scheme is regular).

With the same E and $\mathcal{P}_S$ we show that we have $Q \in \mathcal{P}_S$ (by construction) for which $(E, \mathcal{P}_S)$ is Abel, and the two interpolation systems (of the regular scheme and of the Abel scheme) are equivalent having the same solutions: the $a_{\mathbf{i}} \in \mathbb{R}$. For this we consider the next regrouping $S=\bigcup_{t=1}^{n} {\Delta'}_{k_t}^{d}$ of all indexes of the polinome coefficients P, with $0\le k_1\le k_2\le\ldots\le k_n,\ k_n=\max_{i\in S}|\mathbf{i}|,\ {\Delta'}_{k_t}^{d}\subset\Delta_{k_t}^{d}$ and $\Delta_{k_t}^{d},\ t=\overline{1,n}$, given by (1) (their existence is ensured by proposition 1). Let be:

$$Q(\mathbf{x})=Q(x_1,x_2,\ldots,x_d)=\sum_{\mathbf{i}\in S} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=$$

$$=\sum_{t=1}^{n}Q_{k_t}(\mathbf{x}),Q_{k_t}(\mathbf{x})=\sum_{\mathbf{i}\in S} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}$$

We will determine $b_{\mathbf{i}}$ so that $(E,\mathcal{P}_S)$ is Abel and $b_{\mathbf{i}}=a_{\mathbf{i}}$, for any $\mathbf{i}\in S$. For the beginning let be the system:

$$\sum_{\mathbf{i}\in S,\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}} \tag{4}$$

with $q\in\overline{1,m}$ and $e_{q,\alpha}=1$ for any $\boldsymbol{\alpha}\in S$ and

$$Z=\{\mathbf{x}_q\}_{q=1}^{m}=\{(x_{q,1},x_{q,2}\ldots x_{q,d})\}_{q=1}^{m}\subset\mathbb{R}^d.$$

For $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}_{k_n}^{d}$ and $\boldsymbol{\alpha}\le\mathbf{i}$, $\mathbf{i}\in{\Delta'}_{k_n}^{d}$. In this case the system (4) is equivalent with

$$\sum_{\mathbf{i}\in{\Delta'}_{k_n}^{d},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\ldots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

that is, according to proposition 2, equivalent with the system

$$\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\ b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=\mathbf{c}_{q,\boldsymbol{\alpha}}.$$

We consider in what follows

$$\mathbf{c}_{q,\boldsymbol{\alpha}}=c_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}\overset{def}{=}$$
$$=\alpha_1!\alpha_2!\ldots\alpha_{d-1}![k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\ a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}.$$

It follows that

$$b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})},$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}_{k_n}^{d}$, from where we have

$$\dim P_{{\Delta'}_{k_n}^{d}}=$$
$$=\left|\{\boldsymbol{\alpha}\in(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_n-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in{\Delta'}_{k_n}^{d}\}\right|=\left|{\Delta'}_{k_n}^{d}\right|$$

(each derivative $\boldsymbol{\alpha}\in{\Delta'}_{k_n}^{d}$ of Q is interpolated only once), and

$$Q_{k_n}(\mathbf{x})=\sum_{\mathbf{i}\in{\Delta'}_{k_n}^{d}} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=\sum_{\mathbf{i}\in{\Delta'}_{k_n}^{d}} a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\ldots x_d^{i_d}=P_{k_n}(\mathbf{x})$$

If $\boldsymbol{\alpha}\in{\Delta'}_{k_{n-1}}^{d}$, and $\boldsymbol{\alpha}\le\mathbf{i}$, then $\mathbf{i}\in{\Delta'}_{k_{n-1}}^{d}\cup{\Delta'}_{k_n}^{d}$, and the system (4) becomes successively

$$\sum_{\mathbf{i}\in\Delta'^d_{k_{n-1}}\cup\Delta'^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

$$\sum_{\mathbf{i}\in\Delta'^d_{k_{n-1}},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}+$$
$$+\sum_{\mathbf{i}\in\Delta'^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}}.$$

We take now

$$\mathbf{c}_{q,\boldsymbol{\alpha}}\overset{def}{=}\sum_{\mathbf{i}\in\Delta'^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} a_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}+$$
$$+\alpha_1!\alpha_2!\dots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})}.$$

(for $\mathbf{i}\in\Delta'^d_{k_n}$, $b_{\mathbf{i}}=a_{\mathbf{i}}$) and considering the proposition 2, we obtain the system

$$\alpha_1!\alpha_2!\dots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})]!\,b_{\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})}=$$
$$=\alpha_1!\alpha_2!\dots\alpha_{d-1}![k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})},$$

that is

$$b_{\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1})}$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1}))\in\Delta'^d_{k_{n-1}}$.

It follows that

$$\dim P_{\Delta'^d_{k_{n-1}}}=$$
$$=\left|\{\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_{d-1},k_{n-1}-(\alpha_1+\alpha_2+\dots+\alpha_{d-1}))\in\Delta'^d_{k_{n-1}}\}\right|=\left|\Delta'^d_{k_{n-1}}\right|$$

(each derivative $\boldsymbol{\alpha}\in\Delta'^d_{k_{n-1}}$ of Q is interpolated only once), respectively

$$Q_{k_{n-1}}(\mathbf{x})=\sum_{\mathbf{i}\in\Delta'^d_{k_{n-1}}} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}=\sum_{\mathbf{i}\in\Delta'^d_{k_{n-1}}} a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}=P_{k_{n-1}}(\mathbf{x})$$

Continuing in the same manner, we obtain that for $\boldsymbol{\alpha}\in\Delta'^d_{k_1}$, the system (4) is successively equivalent with

$$\sum_{\mathbf{i}\in\Delta'^d_{k_1}\cup\dots\cup\Delta'^d_{k_n},\boldsymbol{\alpha}\le\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}},$$

$$\sum_{\mathbf{i}\in\Delta'^{d}_{k_1},\boldsymbol{\alpha}\leq\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}+$$

$$+\sum_{\mathbf{i}\in\Delta'^{d}_{k_2}\cup\ldots\cup\Delta'^{d}_{k_n},\boldsymbol{\alpha}\leq\mathbf{i}} b_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}=\mathbf{c}_{q,\boldsymbol{\alpha}}$$

In this final system we take

$$\mathbf{c}_{q,\boldsymbol{\alpha}}\overset{def}{=}\sum_{u=2}^{n}\sum_{\mathbf{i}\in\Delta'^{d}_{k_u},\boldsymbol{\alpha}\leq\mathbf{i}} a_{\mathbf{i}}\frac{i_1!}{(i_1-\alpha_1)!}\cdots\frac{i_d!}{(i_d-\alpha_d)!}x_{q,1}^{i_1-\alpha_1}\dots x_{q,d}^{i_d-\alpha_d}+$$

$$+\alpha_1!\alpha_2!\dots\alpha_{d-1}![k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})]!\,a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}.$$

(for $\mathbf{i}\in\Delta'^{d}_{k_2}\cup\ldots\cup\Delta'^{d}_{k_n}$, $b_{\mathbf{i}}=a_{\mathbf{i}}$).

It follows that in this case we have also

$$b_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}=a_{\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1})}$$

for any $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in\Delta'^{d}_{k_1}$, and thus

$$\dim P_{\Delta'^{d}_{k_1}}=$$

$$=\left|\{\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},k_1-(\alpha_1+\alpha_2+\ldots+\alpha_{d-1}))\in\Delta'^{d}_{k_1}\}\right|=\left|\Delta'^{d}_{k_1}\right|,$$

(each derivative $\boldsymbol{\alpha}\in\Delta'^{d}_{k_1}$ of Q is interpolated only once), respectively

$$Q_{k_1}(\mathbf{x})=\sum_{\mathbf{i}\in\Delta'^{d}_{k_1}} b_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}=\sum_{\mathbf{i}\in\Delta'^{d}_{k_1}} a_{\mathbf{i}}x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}=P_{k_1}(\mathbf{x})$$

In the end, it follows that

$$\sum_{t=1}^{n}\dim P_{\Delta'^{d}_{k_t}}=|S|=\dim\mathcal{P}_S,$$

and

$$Q(\mathbf{x})=\sum_{t=1}^{n}Q_{k_t}(\mathbf{x})=\sum_{t=1}^{n}P_{k_t}(\mathbf{x})=P(\mathbf{x})$$

is the solution of the interpolation problem of the Abel scheme $(E,\mathcal{P}_S)$ (each derivative $\boldsymbol{\alpha}\in S$ of Q is interpolated only once). It follows, according to definition 5, that $(E,\mathcal{P}_S)$ is equivalent to an Abel interpolation scheme q.e.d. □

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