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ROBUST STABILITY OF COMPACT C_0-SEMIGROUPS ON BANACH SPACES

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Abstract. It is a well-known fact that many properties, of a compact semigroup, are preserved under bounded perturbations. In this paper we show that the asymptotic stability is also preserved, provided that the spectral radius of the perturbation is not greater than the modulus of the spectral bound of the semigroup's generator. We achieve our goal by improving Pazy's result concerning the behaviour of the spectrum of the generator.

1. **Preliminaries**

Consider X a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a C_0-semigroup with generator $A : D(A) \subset X \to X$, denoted by $(A, D(A))$.

We use the theoretical notations for $R(\lambda, A)$, $\rho(A)$, $\sigma(A)$, for the resolvent, the resolvent set and, respectively, the spectrum of A. We also use the following notations: the point spectrum $\sigma_p(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not injective}\}$; the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$; the spectral radius $r(A) = \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$.

Let us also remind that the semigroup $\mathcal{T}$ is *asymptotically stable* if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \text{ for any } x \in X.$$

We also use to say that a certain property of the semigroup is *robust* whenever it is preserved under some bounded perturbations.

2. **About the spectrum and robust asymptotic stability of a compact semigroup**

In the following we shall need an auxiliary result from complex analysis, the proof of which is included for the reader's convenience.

Lemma 1. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be a sequence in* $S_{a,b} = \{\lambda \in C \mid a \leq \operatorname{Re}\lambda \leq b\}$, *where* a, b *are real numbers, such that* $\lim_{n\to-\infty} |\operatorname{Im}\lambda_n| = \infty$. *Then there is* $t > 0$ *such that* $\{e^{t\lambda_n}\}_{n=1}^{\infty}$ *has infinitely many acumulation points.*

Proof. We may assume that $0 \leq \operatorname{Im}\lambda_n$ for all $n \geq 1$. Let $J = [0, 1]$ and $\{q_m\}_{m=1}^{\infty}$ be a dense sequence in $[0, 2\pi]$.

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Let $\{A_n\}_{n=1}^{\infty}$ be an enumeration of the sets $B_{m,k} = \{re^{is} \mid s \in q_m + [0, k^{-1}], e^a \leq r \leq e^b\}$, for $k, m \in \mathbb{N}$. The claim is that ther is $t > 0$ and a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that $e^{t\lambda_{n_k}} \in A_k$ for every $k \geq 1$. Clearly the assertion follows from this.

To establish the claim, choose $n_1 \in N$ such that $A_1 \subseteq \{re^{is}; s \in (\operatorname{Im}\lambda_{n_1})J, e^a \leq r \leq e^b\}$. Let $J_1 \subseteq J$ be a closed subinterval with $A_1 = \{re^{is} \mid s \in (\operatorname{Im}\lambda_{n_1})J_1, e^a \leq r \leq e^b\}$. Inductively, we obtain a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\lambda_n\}_{n=1}^{\infty}$ and closed intervals $J \supseteq J_1 \subseteq J_2 \supseteq \cdots$ such that $A_k = \{re^{is} \mid s \in (\operatorname{Im}\lambda_{n_k})J_k, e^a \leq r \leq e^b\}$ for $k \geq 1$. Choose any $t \in \bigcap_{k\geq 1} J_k$. Then $e^{t\lambda_{n_k}} \in A_k$ for all $k \geq 1$.

The following theorem improves in the second part one of Pazy's results [3], and using another approach, also gives a more elementary proof, for the first part of the theorem.

Theorem 2. *Let X be a Banach space and $\mathcal{T} = (T(t))_{t\geq 0}$ a compact C_0-semigroup with generator $(A, D(A))$. Then $\sigma(A)$ consists of a sequence of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, with finite multiplicity, and satisfies* $\lim_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$.

Proof. It is known that $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq w\} \subset \rho(A)$ for some $w \in \mathbb{R}$. As $\rho(A) \neq \emptyset$ choose $\eta \in \rho(A)$ and define $R(\eta, A) \in L(X)$. As $(T(t))_{t\geq 0}$ is a compact semigroup it follows that $R(\eta; A)$ is a compact operator [3] which means that $\sigma(R(\eta; A))$ is a sequence of isolated eigenvalues $\{\eta_n\}_{n=1}^{\infty}$ for $R(\eta; A)$ having 0 as the single accumulation point [2]. By the spectral mapping theorem we have $\sigma(R(\eta, A)) = \{0\} \cup \{(\eta - \lambda)^{-1} \mid \lambda \in \sigma(A)\}$. Since eigenvalues of $R(\eta; A)$ correspond to eigenvalues of $(A, D(A))$ having the same finite multiplicity, then, the first part of our claim follows.

Let us denote by $\{\lambda_n\}_{n=1}^{\infty}$ the sequence of eigenvalues of A.

As $\eta_n = \dfrac{1}{\eta - \lambda_n}$, which means $\lambda_n = \eta - \dfrac{1}{\eta_n}$ it follows that for $\eta_n \to 0$, $\lambda_n \to \infty$ and thus $\{\lambda_n\}_{n=1}^{\infty}$ is an unbounded sequence.

Consider now $S_{a,b} = \{\lambda \in \mathbb{C} \mid a \leq \operatorname{Re}\lambda \leq b\}$ with $a, b \in \mathbb{R}$, $a < b$, and denote by $\{\lambda_{n_k}\}_{k\in N} = S_{a,b} \cap \sigma(A)$.

Suppose that $\{\lambda_{n_k}\}_{k\in N^*}$ is an infinite set. As $\{\lambda_n\}_{n\in N^*}$ is unbounded we deduce that $\lim_{k\to\infty} |\operatorname{Im}\lambda_{n_k}| = \infty$. Then, by Lemma 1, it follows there exists $t_0 > 0$ such that $\{e^{t_0\lambda_{n_k}}\}_{k=1}^{\infty}$ has infinitely many accumulation points. Then the spectral inclusion theorem $e^{t_0\sigma(A)} \subset \sigma(T(t_0))$, and the fact that $\{e^{t_0\lambda_{n_k}}\}_{k\in N^*} \subseteq \{e^{t_0\lambda_n}\}_{n\in N^*}$ imply $\sigma(T(t_0))$ has infinitely many accumulation points. But $T(t_0)$ is a compact operator, therefore it has at most one point of accumulation. That means that $\{\lambda_{n_k}\}_{k\in N^*}$ is always finite for any $a, b \in \mathbb{R}$, $a < b$ and $\lim_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$.

In the following, we shall use a perturbation result from the semigroup theory.

Lemma 3. *Let $(A, D(A))$ be the generator of a C_0–semigroup defined on a Banach space X. If $B \in \mathcal{L}(X)$, then $C = A + B$, where $D(C) = D(A)$, is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$. In addition if $(T(t))_{t\geq 0}$ is compact then $(S(t))_{t\geq 0}$ is also compact.*

Now, let's consider $L(X)$ the space of all linear, bounded operators defined on X.

Theorem 4. *Let X be a Banach space and let $(A, D(A))$ be the infinitesimal generator of an asymptotically stable, compact C_0-semigroup $(T(t))_{t\geq 0}$. If $B \in L(X)$ and B commutes with A, with $r(B) < |s(A)|$, then the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $A + B$ is also asymptotically stable.*

Proof. As $(T(t))_{t\geq 0}$ is stable it follows that it is also bounded and therefore $\sigma(A) \subseteq \{\lambda \in C \mid \operatorname{Re}\lambda \leq 0\}$. By Theorem 2.4 [1], a necessary and sufficient condition for the strong stability of $\mathcal{T} = (T(t))_{t\geq 0}$ is $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. Therefore, as $\sigma(A) = \{\lambda_n\}_{n=1}^{\infty}$ and $\operatorname{Re}\lambda_n < 0$ for any $n \in N^*$ and as $\lim\limits_{n\to\infty} \operatorname{Re}\lambda_n = -\infty$ it follows that $s(A) = \max\limits_{n}\{\operatorname{Re}\lambda_n\} < 0$.

By Lemma 3, the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $A + B$, is also compact. That means that, by the previous theorem, $\sigma(A + B) = \{\mu_n\}_{n\in N^*}$ with $\lim\limits_{n\to\infty} \operatorname{Re}\mu_n = -\infty$ and so $s(A + B) = \max\limits_{n}\{\operatorname{Re}\mu_n\}$.

If $s(A + B) \leq s(A) < 0$, it means that $s(A + B) < 0$ and thus $\sigma_p(A + B) \cap i\mathbb{R} = \emptyset$ and in this case $\mathcal{S}$ is asymptotically stable. Suppose that $s(A+B) > s(A)$. As B commutes with A, a theorem of Kato [2] assures us, that the Pompeiu–Hausdorff distance between $\sigma(A)$ and $\sigma(A + B)$ does not exceed the $r(B)$, $\operatorname{dist}(\zeta, \sigma(A + B)) \leq r(B)$ if $\zeta \in \sigma(A)$.

As $s(A + B) - s(A) \leq \operatorname*{dist}\limits_{\zeta\in\sigma(A)}(\zeta, \sigma(A + B)) \leq r(B) < |s(A)|$ it follows that $s(A + B) < 0$, which proves the theorem.

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