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ATTRIBUTIVE CAUSALITY

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Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

1. Motivation

It is known that each of the notion is characterized by some basic properties and by a set of individuals, satisfying these properties. Both elements mentioned above are expressed by conventional terms.

The judgements, as relations between terms, are formally expressed by propositions as (binary) relations in the set of terms. But the pairing of two terms in a relation supposes new attributes.

Example 1. The proposition 'a is the son of b', near the fact that a, b are human individuals, suggests also new attributes concerning personnel properties and/or mutual obligations (see also Example 3).

2. Algebraical step

If M is a set, then each element $x \in M$ is characterized by a set $\mathcal{A}_x$ of attributes from the universe U of all the attributes. We accept that the set $\mathcal{A}_x$ distinguishes the element from any other element of M. This fact may be formulated by

Axiom 1. $x \neq t \Rightarrow \mathcal{A}_x \neq \mathcal{A}_t$, $\forall x, t \in M$.

Denote by $\mathcal{A}_M$ the set of all the attributes of all the elements of M and observe that

$$\mathcal{A}_M = \bigcup_{x \in M} \mathcal{A}_x.$$

On the other hand, the fact that the elements belong to the same set M offers some common attributes. Therefore we are able to formulate

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Axiom 2. $I_M = \bigcap_{x\in M} \mathcal{A}_x \neq \emptyset$.

Remark 1. $M \subseteq N \Rightarrow I_N \subseteq I_M$; in particular, $I_\emptyset = U$.

Corollary 1. $\forall x \in M : \ \mathcal{A}_x \neq \emptyset$.

Proposition 1. $I_M \cap I_N \subseteq I_{M\cap N}$.

Proof. $I_M \cap I_N = (\bigcap\limits_{x\in M} \mathcal{A}_x) \cap (\bigcap\limits_{y\in N} \mathcal{A}_y) = \bigcap\limits_{z\in M\cup N} \mathcal{A}_z \subseteq \bigcap\limits_{t\in M\cap N} \mathcal{A}_t = I_{M\cap N}$.

We also consider that the name itself of the element x is an attribute of the notion designated by x; this justifies

Axiom 3. $\forall x \in M : \ x \in \mathcal{A}_x$.

Corollary 2. $M \subseteq U$.

Remark 2. $\forall x \in M : \ |\mathcal{A}_x| \geq 2$.

This follows from Axiom 1, Corollary 1 and Axiom 3.

Proposition 2. $M \neq \emptyset \Leftrightarrow \mathcal{A}_M \neq \emptyset$.

Proof. $x \in M \neq \emptyset \Rightarrow x \in \mathcal{A}_x \subseteq \mathcal{A}_M \neq \emptyset$.

$\mathcal{A}_M \neq \emptyset \Rightarrow \exists x \in M : \ \mathcal{A}_x \neq \emptyset \Rightarrow M \neq \emptyset$.

Proposition 3. $\mathcal{A}_{M\cup N} = \mathcal{A}_M \cup \mathcal{A}_N$.

Corollary 3. $M \subseteq N \Rightarrow \mathcal{A}_M \subseteq \mathcal{A}_N$.

Proof. $M \subseteq N \Leftrightarrow M \cup N = N \Leftrightarrow \mathcal{A}_{M\cup N} = \mathcal{A}_N$; but $\mathcal{A}_{M\cup N} = \mathcal{A}_M \cup \mathcal{A}_N$ (Proposition 3), and so $\mathcal{A}_M \cup \mathcal{A}_N = \mathcal{A}_N \Leftrightarrow \mathcal{A}_M \subseteq \mathcal{A}_N$.

Corollary 4. $\mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \cap \mathcal{A}_N$.

Proof. As $M \cap N \subseteq M$ and $M \cap N \subseteq N$, with Corollary 3 it results that:

$$\mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \quad \text{and} \quad \mathcal{A}_{M\cap N} \subseteq \mathcal{A}_N \Rightarrow \mathcal{A}_{M\cap N} \subseteq \mathcal{A}_M \cap \mathcal{A}_N.$$

Remark 3. In Corollary 3, the equality is not true, as it results from:

Example 2. Let M be the set of all triangles in the plane and N the set of squares.

$\mathcal{A}_M = \{\text{triangle, convex, bounded}, \ldots\}$

$\mathcal{A}_N = \{\text{square, convex, bounded}, \ldots\}$

As $M \cap N = \emptyset$ (because there is not 'square–triangle') with Proposition 2 we have $\mathcal{A}_{M\cap N} = \emptyset$; but $\mathcal{A}_M \cap \mathcal{A}_N \neq \emptyset$ (it contains at least the convex and bounded plane figures).

3. Attributive extensions

Given the binary relation $r = (A, B, R)$, the statement $(a, b) \in R \subseteq A \times B$ offers a very dry information concerning individuals a, b as well as the pair (a, b).

Example 3. The relation $r = (A, A, R)$, where A is the set of human individuals, and

$$(x, y) \in R \Leftrightarrow \text{`x is the son of y'}$$

ignore essential attributes such as: the rights or the obligations of x relatively to y, the mutual affection and so on.

From this arises the necessity to consider the corresponding attributive sets $\mathcal{A}_A$, $\mathcal{A}_B$ the attributive extension.

Definition 1. The *attributive extension* of the relation $r = (A, B, R)$ is the relation $\mathfrak{r} = (\mathcal{A}_A, \mathcal{A}_B, \mathcal{R})$, where

$$(\lambda, \pi) \in \mathcal{R} \Leftrightarrow \text{ there is } (a, b) \in R \text{ such that } (\lambda, \pi) \in \mathcal{A}_a \times \mathcal{A}_b.$$

We recall that $s = (C, D, S)$ is a *natural extension* of $r = (A, B, R)$ if $r \subseteq s$, that is $A \subseteq C$, $B \subseteq D$, $R \subseteq S$. In this case, r is a *natural restriction* of s.

Remark 4. If s is a natural extension of r then $\mathfrak{s} = (\mathcal{A}_C, \mathcal{A}_D, \mathcal{S})$ is a natural extension of $\mathfrak{r}$, that is

$$r \subseteq s \Rightarrow \mathfrak{r} \subseteq \mathfrak{s}.$$

This results from Corollary 3.

Proposition 4. Any attributive extension is also a natural extension

$$r \subseteq \mathfrak{r}.$$

Proof. From the Axiom 3 we have:

$$A \subseteq \mathcal{A}_A, \ \ B \subseteq \mathcal{A}_B.$$

From the Definition 1 we obtain:

$$(a, b) \in R, \ \ a \in \mathcal{A}_a \text{ and } b \in \mathcal{A}_b \Rightarrow (a, b) \in \mathcal{R},$$

so $R \subseteq \mathcal{R}$.

The main purpose of this paper is to suggest a distinction between the 'formal' and the 'causative' relations.

Definition 2. The *pair* $(a, b) \in A \times B$ is *causative* if

$$\mathcal{A}_a \cap \mathcal{A}_b \setminus I_A \cap I_B \neq \emptyset \text{ (see also Axiom 2).}$$

Otherwise, the *pair* (a, b) is *formal*.

The *relation* $r = (A, B, R)$ is called *causative* if all the pairs $(a, b) \in R$ are causative. If all the pairs in R are formal, then the *relation* r is called *formal*.

From this point of view, two particular relations are disputed

$$\delta_A = (A, A, \Delta_A) \text{ and } o = (A, B, \emptyset).$$

The principle of identity impose the 'causativity' of the first and the common sense impose the 'formality' of the second. In this light, we formulate

Axiom 4. a) The identical relation δ_A is causative.

b) The empty relation o is formal.

4. **Prospect**

(1) The (two-valued) predicates on the set M may be considered as relations between predicative letters $\mathcal{P} \in \Pi$ and the individuals $x \in M$. The problem is to select these predicates $\mathcal{P}(x)$ for which the pair $(\mathcal{P}, x)$ is causative (see [5]).

(2) The causative relations suggest an 'algebraic refinement' of the social relations between individuals or (professional, confessional) groups (see [3]).

(3) Starting from the correspondence $x \mapsto \mathcal{A}_x$ we may define some 'attributive operations' between sets, which allows us to approach aesthetic problems (see [4]).

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