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ADJOINTS OF LIPSCHITZ MAPPINGS

ŞTEFAN COBZAŞ

Dedicated to Professor Wolfgang W. Breckner at his 60^{th} *anniversary*

Abstract. The aim of this paper is to show that the Lipschitz adjoint of a Lipschitz mapping F, defined by I. Sawashima, Lecture Notes Ec. Math. Syst., Vol. 419, Springer Verlag, Berlin 1975, pp. 247-259, corresponds in a canonical way to the adjoint of a linear operator associated to F.

1. **Introduction**

Let X be a metric space with a distinguished point e (a fixed point in X which is taken to be the zero element if X is a normed space). A metric space X with a distinguished point e is called also a pointed metric space. For a Banach space Y denote by $\mathrm{Lip}_0(X, Y)$ the space of all Lipschitz mappings $F : X \to Y$ vanishing at e. Equipped with the norm

$$L(F) = \sup\{\|F(x_1) - F(x_2)\|/\|x_1 - x_2\| : x_1, x_2 \in X,\ x_1 \neq x_2\}$$

$\mathrm{Lip}_0(X, Y)$ becomes a Banach space. For $Y = \mathbb{R}$ one puts $\mathrm{Lip}_0(X) = \mathrm{Lip}_0(X, \mathbb{R})$. It was shown by Arens and Eels [5] (see also [19]) that $\mathrm{Lip}_0(X)$ is even a dual Banach space, i.e. there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z^*.

Banach spaces of Lipschitz functions, called also Lipschitz duals, were used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case. For instance, Schnatz [18] used them to prove duality and characterization results in best approximation problems in a linear metric space X. In this case one could happen that the dual X^* of X be trivial, $X^* = \{0\}$, so that the methods of linear functional analysis doesn't work. Sawashima [17] defined

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Lipschitz duals of Lipschitz mappings and proved some nonlinear ergodic theorems (see also [14] and [16]).

Lipschitz mappings were also considered in the attempt to develop a nonlinear spectral theory, see [2], [4], [15]. The paper [8] contains a survey on extension results for Lipschitz mappings and their connections to some best approximation problems in spaces of Lipschitz functions.

The aim of this paper is to show that, for a normed space X, the realization of $\mathrm{Lip}_0(X)$ as a dual space can be pushed a little further to obtain a correspondence between Lipschitz duals of Lipschitz mappings and the adjoints of some linear operators. This fact allows to prove some results for Lipschitz mappings by reducing them to the linear case.

2. **The Lipschitz adjoint of a Lipschitz mapping**

We shall present first the construction of Arens and Eels [5] (see also [19, p.38]) of the space for which $\mathrm{Lip}_0(X)$ is the dual space. Remark that another, less explicit, realization of $\mathrm{Lip}_0(X)$ as a dual space was given by de Leeuw [10] (see also [19, p. 33]).

Let (X,ρ) be metric space. A *molecule* on X is a function $m : X \to \mathbb{R}$ with finite support $\sigma(m) = \{x \in X : m(x) \neq 0\}$, and such that $\sum_{x\in X} m(x) = 0$. Denote by $M(X)$ the space of molecules on X. For $x, y \in X$ put $m_{x,y} = h_x - h_y$, where h_x denotes the characteristic function of the set $\{x\}$. One can show that every $m \in M(X)$ can be written, in at least one way, in the form $m = \sum_{i=1}^{n} a_i m_{x_i,y_i}$. Put

$$\|m\|_{AE} = \inf\{\sum_{i=1}^{n} |a_i|\rho(x_i, y_i) : m = \sum_{i=1}^{n} a_i m_{x_i,y_i}\}.$$

It follows that $\| \; \|_{AE}$ is a norm on the vector space $M(X)$. Denote by $AE(X)$ the completion of the normed space $(M(X), \| \; \|_{AE})$. The application $i_X : X \to AE(X)$ defined by

$$i_X(x) = m_{x,e} \tag{1}$$

is an isometric embedding of X into $AE(X)$. Define $S : AE(X)^* \to \mathrm{Lip}_0(X)$ by

$$(S\varphi)(x) = \varphi(m_{x,e}), \qquad \varphi \in AE(X)^*. \tag{2}$$

It follows that S is a nonexpansive linear mapping

$$L(S\varphi) \leq \|\varphi\|, \qquad \varphi \in AE(X)^*.$$

Define now an application $R : \mathrm{Lip}_0(X) \to AE(X)^*$ in the following way. For $f \in \mathrm{Lip}_0(X)$ let first

$$(Rf)(m) = \sum_x m(x)f(x), \quad m \in M(X). \tag{3}$$

Since

$$|(Rf)(m)| \le L(f)\|m\|_{AE}$$

it follows that Rf is a continuous linear functional on $M(X)$, which uniquely extends to a continuous linear functional on the completion $AE(X)$ of $M(X)$, denoted by the same symbol Rf. Therefore $Rf \in AE(X)^*$ and

$$\|Rf\| \le L(f), \quad f \in \mathrm{Lip}_0(X).$$

Straightforward calculations show that R and S are inverses, so that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $AE(X)^*$.

The Banach space $AE(X)$ has some remarkable properties, from which we mention the following one, where the application i_X is defined by (1).

Theorem 1. [19, Theorem 2.2.4] *Let X be a pointed metric space and Y a Banach space. For every $F \in \mathrm{Lip}_0(X,Y)$ there exists a unique continuous linear map $\Psi(F) : AE(X) \to Y$ such that $\Psi(F) \circ i_X = F$. Furthermore $\|\Psi(F)\| = L(F)$.*

From now on we shall suppose that X and Y are real normed spaces, so that the distinguished points are their null elements. Sawashima [17] defined the Lipschitz adjoint (or dual) $F^\# : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ of a Lipschitz map $F \in \mathrm{Lip}_0(X,Y)$ by the formula

$$F^\# g = g \circ F, \quad g \in \mathrm{Lip}_0(Y).$$

He showed that $F^\#$ is a continuous linear operator and that

$$\|F^\#\| = L(F) = \|F^\#|_{Y^*}\|.$$

We shall show that $F^\#$ corresponds in a canonical way to the usual adjoint of the linear operator attached to F by Theorem 1.

Let $F \in \mathrm{Lip}_0(X,Y)$ and let i_X, i_Y be the isometric embeddings of X, Y into $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$, respectively (see (1)). Let $\Psi(F) : AE(X) \to Y$ be the bounded linear operator attached to F by Theorem 1, and let

$$\Phi = i_Y \circ \Psi.$$

Let also S_1, R_1, and S_2, R_2 be the linear isometries between the spaces $\mathrm{Lip}_0(X)$ and $AE(X)^*$, and $\mathrm{Lip}_0(X)$ and $AE(X)^*$, respectively (see the formulae (2) and (3)).

Theorem 2. *We have*

$$F^{\#} = S_1 \circ \Phi(F)^* \circ R_2 \quad \textit{or, equivalently,} \quad \Phi(F)^* = R_1 \circ F^{\#} \circ S_2$$

i.e. the following diagrams are commutative:

$$\begin{array}{ccc} AE(Y)^* & \xrightarrow{\Phi(F)^*} & AE(X)^* \\ {\scriptstyle R_2}\big\uparrow & & {\scriptstyle S_1}\big\downarrow \\ \mathrm{Lip}_0(Y) & \xrightarrow{F^{\#}} & \mathrm{Lip}_0(X) \end{array} \quad \textit{or, equivalently,} \quad \begin{array}{ccc} AE(Y)^* & \xrightarrow{\Phi(F)^*} & AE(X)^* \\ {\scriptstyle S_2}\big\downarrow & & {\scriptstyle R_1}\big\uparrow \\ \mathrm{Lip}_0(Y) & \xrightarrow{F^{\#}} & \mathrm{Lip}_0(X) \end{array}$$

Proof.

We have

$$\Phi(m_{x,0}) = i_Y(\Psi(F)(m_{x,0}) = i_Y(F(x)) = m_{F(x),0}. \tag{4}$$

Put

$$T = S_1 \circ \Phi(F)^* \circ R_2,$$

Therefore

$$\begin{aligned} &(S_1\varphi)(x) = \varphi(M_{x,0}), \quad x \in X,\ \varphi \in AE(X)^*. \\ &\Phi(F)^*(\psi) = \psi \circ \Phi(F), \quad \psi \in AE(Y)^*, \\ &(R_2 g)(m) = \sum_{y \in Y} m(y)g(y), \quad g \in \mathrm{Lip}_0(Y),\ m \in M(Y). \end{aligned}$$

Taking into account these formulae, the definitions of the operators R and S, and formula (4), we obtain successively:

$$\begin{aligned} (Tg)(x) &= (S_1 \circ \Phi(F)^* \circ R_2)(g)(x) = S_1(\Phi(F)^*(R_2 g))(x) \\ &= S_1((R_2 g) \circ \Phi(F))(x) = \\ &= ((R_2 g) \circ \Phi(F))(m_{x,0}) = \\ &= (R_2 g)(m_{x,0}) = g(F(x)) = (g \circ F)(x) = F^{\#}(g)(x). \end{aligned}$$

Theorem 2 is proved. $\square$

We conclude by some open questions. Schauder theorem on the compactness of the adjoint of a compact linear operator between two Banach spaces is well known:

If $A : X \to Y$ is linear and compact then its adjoint $A^* : Y^* \to X^*$ is also compact. In connection with this property we raise the following problems.

Problem 1. Which conditions on a Lipschitz operator $F \in \mathrm{Lip}_0(X, Y)$ entail the compactness of the associated operator $\Phi(F) : AE(X) \to AE(Y)$?

Problem 2. Prove a Schauder type theorem for the Lipschitz adjoint $F^{\#}$ of a Lipschitz operator $F \in \mathrm{Lip}_0(X, Y)$.

Yamamuro [20] defined another kind of adjoint of a Fréchet differentiable mapping and proved a Schauder type theorems for such adjoints. Yamamuro defined the adjoint of a Fréchet differentiable mapping F of a Hilbert space X into itself as a mapping $G : X \to X$ such that $G' = (F')^*$, where A^* denotes the Hilbert adjoint of a continuous linear operator A on X. A thorough study of compactness for nonlinear mappings and their adjoints is done by Batt [6], but his results do not cover the Lipschitz case considered here. Lipschitz duals and duals of Lipschitz mappings were considered in [14, 16] too.

A natural hypothesis for Problem 2 would be the compactness of F, meaning that it sends bounded sets into relatively compact sets. To work with compact sets in $\mathrm{Lip}_0(X, Y)$ we need compactness criteria in spaces of Lipschitz functions. As pointed out J. Appel [1], there are no such criteria, and it turned out that some existing ones were false (e.g. those in [11] or [12]). In this context the following problem is apparently still open:

Problem 3. Find compactness criteria in the space $\mathrm{Lip}_0(X, Y)$.

In [9] we have proved a compactness criterium, but only for families of continuous Fréchet differentiable Lipschitz operators defined on an open subset of a Banach space X and with values in another Banach space Y.

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Babeş-Bolyai University, Faculty of Mathematics and Computer Science, RO-3400 Cluj-Napoca, Romania
E-mail address: scobzas@math.ubbcluj.ro