STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPLICATION OF CLOSE TO CONVEXITY CRITERION TO FILTRATION THEORY

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. We give new and more simple proofs for the univalence of certain functions related to an inverse boundary problem in the theory of filtration. These proofs are based on the criterion of close to convexity.

1. Introduction and statement of the results

Let U be the unit disc of the complex plane $\mathbb{C}$ and let $H(U)$ denote the class of holomorphic functions in U.

A function $f \in H(U)$, with $f(0) = 0$ is called starlike if it is univalent and $f(U)$ is starlike (with respect to the origin). A necessary and sufficient condition for f to be starlike is given by $f'(0) \neq 0$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U.$$

A function $f \in H(U)$ is called convex if it is univalent and $f(U)$ is convex.

A necessary and sufficient condition for f to be convex is given by $f'(0) \neq 0$ and

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

It is easy to show that the function f is convex if and only if the function $g(z) = zf'(z)$ is starlike (Alexander duality theorem).

The function $f \in H(U)$ is called close-to-convex if there exists a convex function φ such that

$$\operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0, \quad z \in U.$$

Received by the editors: 14.10.2002.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. close to convex functions, filtration theory.

According to Alexander duality theorem the function f is close-to-convex if there exists a starlike function g such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U. \tag{1}$$

It is well known that all close to convex functions are univalent [4], [6].

In a mathematical model of the theory of filtration [2], [3], [5] it occurs the problem of finding conditions for univalence of the function F defined in the lower half-plane $\Omega = \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta < 0\}$ by

$$F(\zeta) = G(\zeta) + H(\zeta), \tag{2}$$

with

$$G(\zeta) = \frac{i}{\pi}\sqrt{1-\zeta^2}\int_{-1}^{1} \frac{\varphi(t)dt}{(t-\zeta)\sqrt{1-t^2}} \tag{3}$$

and

$$H(\zeta) = -\frac{2Ti}{\pi} \arctan \frac{\lambda\sqrt{1-\zeta^2}}{\lambda'} = -\frac{T}{\pi} \log \frac{\lambda' + i\lambda\sqrt{1-\zeta^2}}{\lambda' - i\lambda\sqrt{1-\zeta^2}},$$

where $T > 0$, $\lambda, \lambda' \in [0,1]$ with $\lambda^2 + \lambda'^2 = 1$ and

$$0 \le \arg \frac{\lambda' + i\lambda\sqrt{1-\zeta^2}}{\lambda' - i\lambda\sqrt{1-\zeta^2}} \le \pi$$

L. A. Aksentiev in [1] proved the following result by using the argument principle.

Theorem 1. *If the function φ is increasing on $[-1,1]$, then the function F given by (2) is univalent in the half-plane Ω.*

We shall give a more simple proof of this theorem by using the criterion of close to convexity.

In addition we shall prove the following result.

Theorem 2. *If the function φ is increasing on $[-1,1]$, then the function G given by (3) is univalent in the domain $D = \mathbb{C} \setminus [-1,1]$.*

2. **Proof of Theorem 1**

For $z \in U$, let consider the transform

$$\zeta = -i\frac{1+z}{1-z},$$

which maps the unit disc U on the lower half-plane Ω.

The function F becomes

$$f(z) = F[\zeta(z)] = \frac{i}{\pi}\sqrt{1+z^2}\int_{-1}^{1}\frac{1-tz}{(t-\zeta)\sqrt{1-t^2}}\varphi(t)dt$$
$$-i\frac{2T}{\pi}\arctan\left[\frac{\sqrt{2}\lambda}{\lambda'}\frac{\sqrt{1+z^2}}{1-z}\right], \quad z \in U.$$

Since

$$f'(z) = G'(\zeta)\zeta'(z) + H'(\zeta)\zeta'(z),$$

where

$$G'(\zeta) = \frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{1-t\zeta}{(t-\zeta)^2\sqrt{1-t^2}}\varphi(t)dt$$
$$= -\frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{d}{dt}\left[\frac{\sqrt{1-t^2}}{t-\zeta}\right]\varphi(t)dt = \frac{i}{\pi\sqrt{1-\zeta^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-\zeta}d\varphi(t),$$
$$H'(\zeta) = \frac{2iT\lambda\lambda'}{\pi}\cdot\frac{\zeta}{(1-\lambda^2\zeta^2)\sqrt{1-\zeta^2}}$$

and

$$\zeta'(z) = -\frac{2i}{(1-z)^2},$$

we deduce

$$f'(z) = \frac{\sqrt{2}}{\pi\sqrt{1+z^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t+i-(t-i)z}d\varphi(t)$$
$$-\frac{i2\sqrt{2}T\lambda\lambda'}{\pi}\frac{1+z}{[(1-z)^2+\lambda^2(1+z)^2]\sqrt{1+z^2}}.$$

Since

$$(1-z)^2+\lambda^2(1+z)^2 = -i[1-z+i\lambda(1+z)][\lambda+i+(\lambda-i)z],$$

we deduce

$$[\lambda+i(\lambda-i)z]\sqrt{1+z^2}f'(z) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{\lambda+i+(\lambda-i)z}{t+i-(t-i)z}\sqrt{1-t^2}d\varphi(t)$$
$$+\frac{2\sqrt{2}T\lambda\lambda'}{\pi}\frac{1+z}{[1-z+\lambda(1+z)]}. \tag{4}$$

For $\lambda \in [0,1]$ and $t \in [-1,1]$, we have

$$\mathrm{Re}\,\frac{\lambda+i+(\lambda-i)z}{t+i-(t-i)z} > 0, \quad z \in U.$$

On the other hand we have

$$\mathrm{Re}\,\frac{1+z}{1-z+\lambda(1+z)} = \mathrm{Re}\,\frac{1}{\frac{1-z}{1+z}+i\lambda} > 0, \quad z \in U.$$

Hence for $\lambda \in [0,1]$, from (4) we deduce

$$\mathrm{Re}\,\{[\lambda + i + (\lambda - i)z]\sqrt{1+z^2}f'(z)\} > 0, \quad z \in U. \tag{5}$$

Let the function g be defined by

$$g(z) = \frac{z}{[\lambda + i + (\lambda - i)z]\sqrt{1+z^2}}.$$

Since

$$\frac{zg'(z)}{g(z)} = 1 - \frac{(\lambda - i)z}{\lambda + i + (\lambda - i)z} - \frac{z^2}{1+z^2},$$

and

$$\mathrm{Re}\,\frac{z^2}{1+z^2} < \frac{1}{2}, \quad z \in U$$

$$\mathrm{Re}\,\frac{(\lambda - i)z}{\lambda + i + (\lambda - i)z} = \mathrm{Re}\,\frac{kz}{1+kz} < \frac{1}{2}, \quad z \in U,$$

with $|k| = |(\lambda - i)/(\lambda + i)| = 1$, we deduce

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} > 0, \quad z \in U,$$

which shows that g is starlike. Since (5) can be rewritten as the inequality (1), we deduce that f is close to convex, hence f is univalent in U and this implies that F is univalent in Ω.

3. **Proof of Theorem 2**

For $z \in U$, let consider the Jukowski transform

$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

which maps the unit disc U onto the domain $D = \mathbb{C} \setminus [-1,1]$.

The function G, given by (3) becomes

$$g(z) = G[\zeta(z)] = \frac{1}{\pi}(1 - z^2)\int_{-1}^{1} \frac{\varphi(t)dt}{(1 - 2tz + z^2)\sqrt{1-t^2}}.$$

Since for $t = \cos\theta$, we have

$$\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} = \frac{1 - z^2 - 2iz\sin\theta}{1 + z^2 - 2z\cos\theta},$$

and we deduce

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\cos\theta)\left[\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} + 2i\frac{z\sin\theta}{1 + z^2 - 2z\cos\theta}\right]d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \varphi(\cos\theta) d\theta.$$

Since the function $\varphi(\cos\theta)$ is increasing on $[-\pi, 0]$ and decreasing on $[0, \pi]$, by applying the well known theorem of Kaplan concerning the Poisson integral [4], we deduce that g is univalent in U and this implies that G is univalent on D.

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