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ON BLEIMANN, BUTZER AND HAHN TYPE GENERALIZATION OF BALÁZS OPERATORS

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. In this paper we introduced a generalization of Balázs operators [4] which includes the Bleimann, Butzer and Hahn operators [6]. We define a space of general Lipschitz type maximal functions and obtain the approximation properties of these operators. Also we estimate the rate of convergence of these operators. In the last section, we obtain derivative and bounded variation properties of these generalized operators.

1. Introduction

In [4], K. Balázs introduced the discrete linear positive operators defined by

$$(R_n f)(x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f(\frac{k}{b_n}) \binom{n}{k} (a_n x)^k, \quad x \geq 0, \ n \in \mathbb{N} \tag{1}$$

where a_n and b_n are positive numbers, independent of x.

After simple computation, we have

$$(R_n e_0)(x) = 1$$

$$(R_n e_1)(x) = \frac{n}{b_n} \frac{a_n x}{1+a_n x}$$

$$(R_n e_2)(x) = \frac{n(n-1)}{b_n^2} \left(\frac{a_n x}{1+a_n x} \right)^2 + \frac{n}{b_n^2} \frac{a_n x}{1+a_n x}$$

where e_n represents the monomial $e_n(x) = x^n$ for $n = 0, 1, 2$.

These equalities show that both of classical Bohman-Korovkin theorems in [7], [13] and weighted Korovkin type theorems in [10] and [9] do not valid.

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In [4], Voronoskaja type formula was given for operators (1), under the some restriction of sequences a_n and b_n.

In [1] and [2], O. Agratini introduced a Kantorovich type integral form of operators (1) and obtained the degree of approximation in polynomial weighted function spaces.

By choosing $a_n = n^{\beta-1}$, $b_n = n^{\beta}$ for $n \in \mathbb{N}$ and $0 < \beta < 1$, the operator (1) was denoted by the symbol $R_n^{[\beta]}$. Also, for some $0 < \beta < 1$ values in [4], [5] and [17], convergence, derivative and saturation properties of $R_n^{[2/3]}$ were investigated by K. Balázs, J. Szabados and V. Totik respectively.

A recent paper is given by O. Agratini in [2] about Voronovskaja type theorem for Kantorovich type generalization of the $R_n^{[\beta]}$.

On the other hand in [6], G. Bleimann, P.L. Butzer and L. Hahn introduced the Bernstein type sequence of linear positive operator defined as

$$(L_n f)(x) = (1+x)^{-n} \sum_{k=0}^{n} f(\frac{k}{n-k+1}) \binom{n}{k} x^k, \quad x \geq 0, \ n \in \mathbb{N}. \tag{2}$$

In [6], pointwise convergence properties of operators (2) are investigated on compact subinterval $[0, b]$ of $[0, \infty)$. In [11], T. Hermann investigated the behavior of operators (2) when the growth condition for f is weaker than polynomial one. In [12], C. Jayasri and Y. Sitaraman proved direct and inverse theorems of operators (2) in the some subspaces on positive real axis. In [8], by using the test functions $\left(\frac{x}{1+x}\right)^{\nu}$ for $\nu = 0, 1, 2$, a Korovkin type theorem was given by Ö. Çakar and A.D. Gadjiev and they obtained some approximation properties of (2) in a subclass of continuous and bounded functions on all positive semi-axis.

The aim of this paper is to investigate the approximation properties of a generalization of K. Balázs's operators R_n in Bleimann, Butzer and Hahn operators type on the all positive semi-axis.

2. **Construction of the operators**

We consider the sequence of linear positive operators

$$(A_n f)(x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_{n,k}}\right) \binom{n}{k} (a_n x)^k, \quad x \geq 0, \ n \in \mathbb{N} \tag{3}$$

where a_n and $b_{n,k}$ satisfy the following conditions for every n and k;

$$a_n k + b_{n,k} = c_n \text{ and } \frac{n}{c_n} \to 1 \text{ for } n \to \infty. \tag{4}$$

Since replacing b_n by $b_{n,k}$, these operators different from the operators R_n.

Clearly, if we choose $a_n = 1$, $b_{n,k} = n - k + 1$ for every n and k then $c_n = n + 1$ the conditions (4) are satisfied. These operators turn out into Bleimann, Butzer and Hahn operators. Therefore, these operators are a Bleimann, Butzer and Hahn operators type generalization of Balázs operators.

3. **Approximation properties**

In this section, we will give a Korovkin type theorem in order to obtain approximation properties of operators (3).

In [14], B. Lenze introduced a Lipschitz type maximal function as

$$f_\alpha^\sim(x) = \sup_{\substack{t>0 \\ t\neq x}} \frac{|f(t) - f(x)|}{|x-t|^\alpha}.$$

Firstly, we define a space of general Lipschitz type maximal functions.

Let $W_\alpha^\sim$ be the space of functions defined as

$$W_\alpha^\sim = \left\{ f : \sup(1+a_n t)^\alpha \; f_\alpha(x,t) \leq M \left(\frac{a_n}{1+a_n x} \right)^\alpha, x \geq 0 \right\} \tag{5}$$

where f is bounded and continuous on $[0, \infty)$, M is a positive constant, $0 < \alpha \leq 1$ and f_α is the following function

$$f_\alpha(x,t) = \frac{|f(t) - f(x)|}{|x-t|^\alpha}.$$

Example 1. For any $M_1 > 1$, let the sequence of functions f_n be

$$f_n(x) = \frac{1 + M_1 a_n x}{1 + a_n x}.$$

Then for all $x, t \geq 0$, $x \neq t$, we have

$$|f(t) - f(x)| = \frac{(M_1 - 1)a_n |x-t|}{(1+a_n x)(1+a_n t)}.$$

By choosing $M_1 - 1 \leq M$, one obtains $f_n \in W_1^\sim$.

Also, if $\dfrac{a_n}{1+a_n x}$ is bounded then $W_\alpha^\sim \subset Lip_{M_1}(\alpha)$ where M_1 is a positive constant which satisfies the following inequality

$$M \left(\frac{a_n}{1+a_n x} \right)^\alpha \left(\frac{1}{1+a_n t} \right)^\alpha \leq M_1.$$

Really, if $f \in W_\alpha^\sim$ then for all $x, t \geq 0$, $x \neq t$ we can write

$$|f(t) - f(x)| \leq M \left(\frac{a_n}{1 + a_n x} \right)^\alpha \left(\frac{1}{1 + a_n t} \right)^\alpha |x - t|^\alpha$$

and $f \in Lip_{M_1}(\alpha)$. Clearly that, if $a_n \leq 1$ or $x \geq 1$ then $\dfrac{a_n}{1 + a_n x}$ is bounded.

Theorem 2. *If L_n is the sequence of positive linear operators acting from $W_\alpha^\sim$ to $C_B[0, \infty)$ and satisfying the following conditions for $\nu = 0, 1, 2$*

$$\left\| \left(L_n \left(\frac{a_n t}{1 + a_n t} \right)^\nu \right)(x) - \left(\frac{a_n x}{1 + a_n x} \right)^\nu \right\|_{C_B} \to 0 \text{ for } n \to \infty \tag{6}$$

then, for any function f in $W_\alpha^\sim$ one has

$$\|L_n f - f\|_{C_B} \to 0 \quad \text{for} \quad n \to \infty.$$

where $C_B[0, \infty)$ denotes the space of functions which is bounded and continuous on $[0, \infty)$.

Proof. This proof is similar to the proof of Korovkin theorem.

Let $f \in W_\alpha^\sim$,. Since f is continuous on $[0, \infty)$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(t) - f(x)| < \epsilon \text{ for } \left| \frac{a_n t}{1 + a_n t} - \frac{a_n x}{1 + a_n x} \right| < \delta$$

and since f is bounded on $[0, \infty)$, there is a positive constant M such that

$$|f(t) - f(x)| < \frac{2M}{\delta^2} \left[\frac{a_n(t - x)}{(1 + a_n t)(1 + a_n x)} \right]^2 \text{ for } \left| \frac{a_n t}{1 + a_n t} - \frac{a_n x}{1 + a_n x} \right| \geq \delta.$$

Thus, for all $t, x \in [0, \infty)$ one has

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2} \left[\frac{a_n(t - x)}{(1 + a_n t)(1 + a_n x)} \right]^2. \tag{7}$$

By using basic properties of positive linear operators, we have

$$\begin{aligned} \|L_n f - f\|_{C_B} &\leq \|(L_n |f - f(x)|)\|_{C_B} + \\ &+ \|f\|_{C_B} \|(L_n 1) - 1\|_{C_B} \end{aligned} \tag{8}$$

By using the inequality (7) and conditions (6) in (8), the proof is complete.

Now, we will give the first main result about approximation properties of operators (3) with the help of Theorem 2.

Theorem 3. *If A_n is the sequence of positive linear operators defined by (3), then for each $f \in W_{\alpha}^{\sim}$*

$$\|(A_n f) - f\|_{C_B} \to 0 \ \ for \ \ n \to \infty.$$

Proof. For the operators in (3), it is easily to verify that

$$(A_n 1)(x) = 1$$
$$\left(A_n\left(\frac{a_n t}{1+a_n t}\right)\right)(x) = \frac{n}{c_n}\frac{a_n x}{1+a_n x}$$
$$\left(A_n\left(\frac{a_n t}{1+a_n t}\right)^2\right)(x) = \left(\frac{n}{c_n}\right)^2\left(\frac{a_n x}{1+a_n x}\right)^2 + \frac{1}{c_n}\frac{n}{c_n}\frac{a_n x}{1+a_n x}.$$

By using the conditions (4) and Theorem 2, the proof is obvious.

4. **Approximation order**

In this section, we give a result about rate of convergence of operators (3).

Theorem 4. *If $f \in W_{\alpha}^{\sim}$ then for all $x \geq 0$ we have*

$$|(A_n f)(x) - f(x)| \leq M\left(\frac{n}{c_n}a_n - 1\right)^{\alpha} \tag{9}$$

where the constants M and $0 < \alpha \leq 1$ are defined in the definition of the space $W_{\alpha}^{\sim}$ and the operators A_n are defined in (3).

Proof. If $f \in W_{\alpha}^{\sim}$, we can write

$$|(A_n f)(x) - f(x)|$$
$$\leq M\left(\frac{a_n}{1+a_n x}\right)^{\alpha}\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\left|\frac{k}{b_{n,k}} - x\right|^{\alpha}\left(\frac{1}{1+a_n\frac{k}{b_{n,k}}}\right)^{\alpha}\binom{n}{k}(a_n x)^k.$$

From the conditions (4), we get

$$\frac{1}{1+a_n\dfrac{k}{b_{n,k}}} = \frac{b_{n,k}}{c_n}.$$

If we use this equality in the last inequality, we obtain

$$|(A_n f)(x) - f(x)|$$
$$\leq M\left(\frac{a_n}{1+a_n x}\right)^{\alpha}\frac{1}{c_n^{\alpha}}\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}|k - x(c_n - a_n k)|^{\alpha}\binom{n}{k}(a_n x)^k.$$

By using the Hölder inequality for $p = \dfrac{2}{\alpha}$, $q = \dfrac{2}{2-\alpha}$ and considering $((A_n e_0)(x))^{\frac{2-\alpha}{2}} = 1$ we have

$$|(A_n f)(x) - f(x)| \tag{10}$$
$$\leq M\left(\frac{a_n}{c_n(1+a_n x)}\right)^{\alpha}\left(\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}(k - x(c_n - a_n k))^2\binom{n}{k}(a_n x)^k\right)^{\frac{\alpha}{2}}.$$

On the other hand, it is obvious that

$$\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k = (1+a_n x)^n,$$

$$\sum_{k=1}^{n}k\binom{n}{k}(a_n x)^k = n a_n x(1+a_n x)^{n-1},$$

$$\sum_{k=1}^{n}k^2\binom{n}{k}(a_n x)^k = (a_n x)^2 n(n-1)(1+a_n x)^{n-2} + a_n x n(1+a_n x)^{n-1}.$$

By using these equalities, after simplifications, we obtain

$$\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}(k - x(c_n - a_n k))^2\binom{n}{k}(a_n x)^k \leq$$
$$\leq x^2 c_n^2\left[\left(\frac{n}{c_n}a_n - 1\right)^2 - \frac{n}{c_n^2}a_n^2\right] \leq x^2 c_n^2\left(\frac{n}{c_n}a_n - 1\right)^2.$$

If we use last inequality in (10), we have

$$\begin{aligned}|(A_n f)(x) - f(x)| &\leq M\left(\frac{a_n}{c_n(1+a_n x)}\right)^{\alpha} x^{\alpha} c_n^{\alpha}\left(\frac{n}{c_n}a_n - 1\right)^{\alpha} \\ &= M\left(\frac{a_n x}{1+a_n x}\right)^{\alpha}\left(\frac{n}{c_n}a_n - 1\right)^{\alpha}.\end{aligned}$$

Since $\left(\dfrac{a_n x}{1+a_n x}\right)^{\alpha} \leq 1$, the proof is complete.

Since Theorem 4 is valid for all $x \geq 0$, this proof gives uniform convergence of the operators A_n to f without using Korovkin type theorem.

5. **Derivative properties**

Firstly, explicit formula for derivatives of Bernstein polynomials with difference operators is obtained by G.G. Lorentz in [15, p.12]. A lot of studies have included derivative properties of positive linear operators. In [16], D.D. Stancu obtained the monotonicity properties from different orders of the derivatives of Bernstein polynomials with the help of divided differences.

In this part, we will give some derivative properties of operators A_n defined in (3) with the help of difference operators.

We can easily compute:

$$\frac{d}{dx}(A_nf)(x) = na_n(1+a_nx)^{-n-1}\sum_{k=0}^{n-1}\left[f(\frac{k+1}{b_{n,k}})-f(\frac{k}{b_{n,k}})\right]\binom{n-1}{k}(a_nx)^k \quad (11)$$

and by using induction method for derivatives of $k-$order, we have

$$\frac{d^k}{dx^k}(A_nf)(x) \quad (12)$$

$$= n(n-1)\dots(n-k+1)a_n^k(1+a_nx)^{-n-k}\sum_{\nu=0}^{n-k}\Delta^kf(\frac{\nu}{b_{n,k}})\binom{n-k}{\nu}(a_nx)^\nu,$$

where $\Delta^kf\left(\frac{\nu}{b_{n,k}}\right)$ is difference operator defined as

$$\Delta^kf(\frac{\nu}{b_{n,k}}) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f\left(\frac{\nu+i}{b_{n,k}}\right).$$

Theorem 5. *Let $f\in C^1[0,\infty)$. Then the operators A_n have the monotonicity properties.*

Proof. If $f\in C^1[0,\infty)$ then $f\in C^1\left[\frac{k}{b_{n,k}},\frac{k+1}{b_{n,k}}\right]$. Therefore, from (11) we can write

$$\frac{d}{dx}(A_nf)(x) = na_n(1+a_nx)^{-n-1}\sum_{k=0}^{n-1}\int_{\frac{k}{b_{n,k}}}^{\frac{k+1}{b_{n,k}}}f'(\xi)d\xi\binom{n-1}{k}(a_nx)^k.$$

Since $\int_{\frac{k}{b_{n,k}}}^{\frac{k+1}{b_{n,k}}}f'(\xi)d\xi\geq 0\ (\leq 0)$ for $f'(x)\geq 0\ (\leq 0)$, we have

$$\frac{d}{dx}(A_nf)(x)\geq 0(\leq 0)\quad\text{for}\quad f'(x)\geq 0(\leq 0)$$

and this completes the proof.

In [15, p.23], G.G. Lorentz gives an estimate related to the total variation of Bernstein polynomials. Similarly, in the following theorem, we give an estimate of bounded variation between the operators A_n and f.

Theorem 6. *The operators A_n preserve the functions of bounded variation on $[0,\infty)$.*

Proof. By using formula (11), we get

$$V_n(A_nf) = \int_0^\infty \left|\frac{d}{dx}(A_nf)(x)\right| dx \tag{13}$$
$$\leq \sum_{k=0}^{n-1} \left|\Delta f(\frac{k}{b_{n,k}})\right| na_n \binom{n-1}{k} \int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx.$$

Since $k > -1$ and $-k+n > 0$, we can write

$$\int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx = \frac{\Gamma(1+k)\Gamma(-k+n)}{a_n\Gamma(1+n)}.$$

If we use properties of Gamma function in this equality, we have

$$\int_0^\infty (a_nx)^k(1+a_nx)^{-n-1}dx = \frac{k!(n-k-1)!}{a_nn!}.$$

By using this equality in (13), we obtain

$$V_n(A_nf) \leq V(f)$$

which gives the proof.

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