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FIXED POINTS OF R-CONTRACTIONS

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. Let X be a set and $R = (R_n)_{n\geq 0}$, $R_n \subset X \times X$ a sequence of binary relations on X. The operator $f : X \longrightarrow X$ is R-contraction if

$$(x, y) \in R_n \Longrightarrow (f(x), f(y)) \in R_{n+1}.$$

The first theorem concerning R-contraction is due to Eilenberg [2]. Further I. A. Rus [7] and Grudzinski [3] generalize this concept. We prove some results which generalize the theorems in [7] and [3] under certain conditions.

1. Introduction

Let X be a set, $f : X \longrightarrow X$ an operator and F_f be a fixed point set of f:

$$F_f := \{x \in X \mid f(x) = x\}.$$

We introduce the following notations:

$$\Delta(X) := \{(x, x) \mid x \in X\},$$

$$f^0 = 1_X,\ f^1 = f,\ f^n(x) := f(f^{n-1}(x)), \quad n \geq 2.$$

Let X be a nonempty set, $R_n \subset X \times X$ a sequence of symmetric binary relations on X. Throughout this paper we suppose that:

a)

$$X \times X = R_0 \supset R_1 \supset ... \supset R_n \supset ...$$

b)

$$\bigcap_{n=0}^{\infty} R_n = \Delta(x) = \{(x, x) \,|\, x \in X\}.$$

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Firstly Eilenberg [2] proved the discrete version of Banach fixed point theorem. Later I. A. Rus [7] introduced the concept of R-contractions:

Definition 1. The operator $f : X \longrightarrow X$ is R-contraction if

$$(x, y) \in R_n \Longrightarrow (f(x), f(y)) \in R_{n+1}.$$

I. A. Rus [7], [6], [8] and indepently I. A. Grudzinsky [3] proved fixed point theorems for R-contractions (see Bege [1]).

In this paper we generalize the concept of R-contractions and we prove some fixed point theorems for this contractions.

2. **Generalized R-contractions**

In this section we introduce the concept of generalized R-contraction and give some examples.

Definition 2. Let $X \neq \emptyset$, $R_n \subset X \times X$, $\quad n \in \mathbb{N}$. We say that $f : X \longrightarrow X$ **generalized R-contraction of the type** d_i if

d$_1$) $(x, f(x)) \in R_n, \quad (y, f(y)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_2$) $(x, y) \in R_n,\ (x, f(x)) \in R_n, \quad (y, f(x)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_3$) $(x, y) \in R_n, \quad (x, f(y)) \in R_n$

$$\Rightarrow (f(x), f(y)) \in R_{n+1}$$

d$_4$)

$$(x, f(x)) \in R_n \Rightarrow (f(x), f^2(x)) \in R_{n+1}$$

We remark that if an operator is R-contraction then it is a generalized d_4 contraction.

In the following part of this section we present some examples concerning R-contractions and generalized contractions.

Example 1 (S. Reich [5])

Let (X, d) be a metric space and $a, b, c \in \mathbb{R}_+,\ a + b + c < 1$ such that

$$d\left(f(x), f(y)\right) \leq a \cdot d\left(x, y\right) + b \cdot d\left(x, f(x)\right) + c \cdot d\left(y, f(y)\right), \quad \forall x, y \in X.$$

If

$$R_n = \left\{(x, y) \in X \times X \mid d(x, y) \leq \frac{a+b}{1-c} \cdot (a+b+c)^n \cdot \delta(X)\right\},$$

$$Y = \left\{x \in X \mid d(x, f(x)) \leq \frac{a+b}{1-c} \cdot \delta\left(X\right)\right\} \neq \emptyset$$

then R_n satisfies the conditions (a) and (b) and f generalized contraction of the type d_2.

Example 2 (R. Kannan [4])

Let (X, d) be a metric space, and $f : X \longrightarrow X$ one operator for which exist $a \in \mathbb{R},\ a < \frac{1}{2}$, such that:

$$d(f(x), f(y)) \leq a \cdot [d(x, f(x)) + d(y, f(y))], \quad \forall x, y \in X.$$

If

$$R_n = \left\{(x, y) \in X \times X \mid d(x, y) \leq \frac{a}{1-a} \cdot (2a)^n \cdot \delta(X)\right\},$$

$$Y = \left\{x \in X \mid d(x, f(x)) \leq \frac{a}{1-a} \cdot \delta\left(X\right)\right\} \neq \emptyset,$$

then R_n satisfies the conditions $a)$ şi $b)$ and f generalized R-contraction of the type d_1.

3. **Main results**

Theorem 3. *Let X be a nonempty set, $R_n \subset X \times X$ a sequence of symmetrical binary relations on X, satisfying the conditions* **a) b)** *and*

c) *If $(x_n)_{n\geq 0}$ is a sequence in X such that $(x_n, x_{n+k}) \in R_n, \quad \forall n, k \geq 0$, then there exist unique $x \in X$ satisfying the condition $(x_n, x) \in R_n, \quad \forall n \geq 0$.*

Let $f : X \longrightarrow X$ be a generalized R-contraction of type d_3. Then f has an unique fixed point.

Proof.

Let $x_0 \in X\ ,\ x_n = f\left(x_{n-1}\right),\ \forall n \geq 1.$

From the form of R_0 and definition 2 we have:

$$(x_0, x_1) \in R_0,\ (x_0, x_2) \in R_0 \Longrightarrow (x_1, x_2) = (f(x_0), f(x_1)) \in R_1,$$

$$(x_0, x_2) \in R_0,\ (x_0, x_3) \in R_0 \Longrightarrow (x_1, x_3) = (f(x_0), f(x_2)) \in R_1.$$

From mathematical induction follows that: $(x_1, x_{n+1}) \in R_1,\ \forall n \geq 0$.
But

$$(x_1, x_n) \in R_1,\ (x_1, x_{n+1}) \in R_1 \Longrightarrow (x_2, x_{n+1}) \in R_2, \quad \forall n \geq 1$$

and generally

$$(x_k, x_{k+n}) \in R_k, \quad \forall k \geq 0, \forall n \geq 0.$$

Condition **c**$_1$**)** implies the existence of unique $x^* \in X$ such that $(x^*, x_n) \in R_n,\ \forall n \geq 0$.
But

$$(x^*, x_n) \in R_n,\ (x^*, x_{n+1}) \in R_{n+1} \subset R_n \Longrightarrow (f(x^*), x_{n+1}) \in R_{n+1},\ \forall n \geq 0.$$

Because x^* is unique, $x^* = f(x^*)$.
If we have $y^* \in X$, for which $y^* = f(y^*)$, then

$$(x^*, y^*) = (x^*, f(y^*)) \in R_0 \Longrightarrow (f(x^*), f(y^*)) = (x^*, y^*) \in R_1$$

Similarly $(x^*, y^*) \in R_n$ for all n.
From **b)** we have $x^* = y^*$.

Corollary 4. ([7], Theorem 2.1) *If $f : X \longrightarrow X$ is a R-contraction, and $R_n \subset X \times X$, $n \in N$, a sequence of binary symmetrical relations, satisfying the conditions* **a) b)** *and* **c)** *, then:*

$$F_f = \{x^*\}$$

and

$$(f^n(x_0), x^*) \in R_n, \quad \forall x_0 \in X,\ n \in N.$$

Theorem 5. *Let X be a nonempty set and $R_n \subset X \times X$, $n \in N$ a sequence of symmetrical binary relations on X, satisfying the conditions* **a)**, **b)**,
c$_1$*)*
If $(x_n)_{n \geq 0}$ is a sequence in X such that $(x_n, x_{n+k}) \in R_n$ for all $n,\ k \in N$ then there

exist unique $x \in X$ *for which* $(x_n, x) \in R_n$, $\forall n \in N$.

If $f : X \longrightarrow X$ *is a generalized R-contraction of type* $\mathbf{d}_1$*) and satisfies the following condition:*

e)

For every $x_0 \in X$

$$\left(f^n(x_0), x\right) \in R_n \Longrightarrow \left(f^{n+1}(x_0), f(x)\right) \in R_{n+1} \quad (n \in N).$$

Then f *has an unique fixed point.*

Proof.

In same way (see the proof of theorem 1) we have that if $x_0 \in X$, $x_n = f(x_{n-1})$, $\forall n \geq 1$ then:

$$(x_k, x_{k+n}) \in R_k, \ \forall k \geq 0, \ \forall n \geq 0.$$

The condition $\mathbf{c}_1$) implies the existence of the unique $x^* \in X$ such that $(x^*, x_n) \in R_n, \ \forall n \geq 0$.

But from **e)** :

$$(x_n, x^*) = \left(f^n(x_0), x^*\right) \in R_n \Longrightarrow \left(f^{n+1}(x_0), f(x^*)\right) = (x_{n+1}, f(x^*)) \in R_{n+1}.$$

We have $(x_0, f(x^*)) \in R_0$ so $(x_n, f(x^*)) \in R_n$ for all n. The uniqueness of x^* implies $x^* = f(x^*)$.

In the next we prove the uniqueness of the fixed point:

Let $x^*, y^* \in F_f$. From **b)** $(x^*, f(x^*)) \in R_n$ and $(y^*, f(y^*)) \in R_n$ for all $n \geq 0$. This implies that $(x^*, y^*) \in R_n$ (f generalized R-contraction of type $\mathbf{d}_1$)). So $x^* = y^*$.

Theorem 6. *Let* X *be a nonempty set and* $R_n \subset X \times X$, $n \in N$ *a sequence of symmetrical binary relations on* X*, satisfying the conditions* **a)**, **b)**,

$\mathbf{c}_2$*)*

If $(x_n)_{n \geq 0}$ *is a sequence in* X *such that* $(x_n, x_{n+k}) \in R_n$ *for all* n, $k \in N$ *then there exist* $x \in X$ *(not necessary unique) for which* $(x_n, x) \in R_n$, $\forall n \in N$.

f) *For all* $x, y, z \in X, \ n \in N$

$$(x, y) \in R_{n+k}, \quad (y, z) \in R_{n+k} \Longrightarrow (x, z) \in R_n.$$

If $f : X \longrightarrow X$ *is a generalized R-contraction of type* $\mathbf{d}_3$*) then* $F_f = \{x^*\}$.

Proof.

We consider the iterares of f in x_0: $x_n = f(x_{n-1}), \quad \forall n \geq 1.$

From the first part of the proof of Theorem 1, there exist $x^* \in X$ such that

$$(x^*, x_{n+k}) \in R_n \quad \forall n \geq 0.$$

f generalized R-contraction of type $\mathbf{d}_3)$ which implies:

$$(x^*, x_{n+2k}) \in R_{n+k}, \ (x^*, x_{n+2k+1}) \in R_{n+k+1} \subset R_{n+k} \Longrightarrow$$

$$\Longrightarrow (f(x^*), x_{n+2k+1}) \in R_{n+k+1} \subset R_{n+k}.$$

From condition **f)**:

$$(x^*, x_{n+2k+1}) \in R_{n+k}, \quad (f(x^*), x_{n+2k+1}) \in R_{n+k} \Longrightarrow (x^*, f(x^*)) \in R_n,$$

$$(x^*, f(x^*)) \in \bigcap_{n \in N} R_n = \Delta(x)$$

which implies $x^* = f(x^*)$.

The proof of uniqueness is same with the proof in Theorem 1.

Corollary 7. (Grudzinski [3]) *Let X be a nonempty set and $R_n \subset X \times X$, $n \in N$ a sequence of reflexive and symmetrical binary relations on X, satisfying the conditions* **a), b), c$_2$), f)** . *Let $f : X \longrightarrow X$ be R-contraction.*

Then f has an unique fixed point.

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