STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 4, December 2002

APPROXIMATION PROPERTIES OF A BIVARIATE STANCU TYPE OPERATOR

DAN BĂRBOSU

Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. An extension of Stancu's operator $P_m^{(\alpha,\beta)}$ to the case of bivariate functions is presented and some approximation properties of this operator are discussed.

1. **Preliminaries**

In 1969 (see[8]), D.D. Stancu constructed and studied a linear and positive operator, depending on two positive parameters α and β which satisfy the condition $0 \le \alpha \le \beta$. This operator, denoted by $P_m^{(\alpha,\beta)}$, associates to any function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$, defined by:

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum\nolimits_{k=0}^{m} p_{mk}(x)\ f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.1}$$

where $p_{mk}(x)$ are the fundamental Bernstein polynomials. In the monograph by F. Altomare and M. Campiti ([1]) this operator is called "the operator of Bernstein-Stancu".

A first extensions of the operator (1.1) to the case of bivariate functions was given by F. Stancu in her doctoral thesis (see [9]). The aim of the present paper is to extend the operator (1.1) to the case of B-continuous (Bőgel continuous functions). More exactly, we shall present a GBS (Generalized Boolean Sum) operator of Stancu type and some properties of this operator.

Received by the editors: 17.09.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A36, 41A63, 41A80.

Key words and phrases. Stancu's operators, Korovkin theorem, bivariate function, modulus of smoothness, Voronovskaia theorem.

The terminus of "B-continuous function" was introduced by K. Bőgel ([5],[6]). A first result concerning the approximation of this kind of functions is due to E. Dobrescu and I. Matei ([7]).

An important "test function theorem", (the analogous of the well known Korovkin theorem), for the approximation of B-continuous functions by GBS operators was introduced by C. Badea and C. Cottin ([3]). Approximation properties of the GBS operators were studied by C. Badea, C. Cottin, H.H. Gonska, D. Kacsó and many others.

2. **The GBS operator of Stancu type**

Let be $I=[0,1]$ and let $I^2=[0,1]\times[0,1]$ be the unit square. The space of all B-continuous functions on I^2 will be denoted by $C_b(I^2)$.

Next, we consider four non-negative parameters $\alpha_1,\beta_1,\alpha_2,\beta_2$,satisfying the conditions $0\le\alpha_1\le\beta_1, 0\le\alpha_2\le\beta_2$. If $f\in C_b(I^2)$, the parametric extensions of the operator $P_m^{(\alpha,\beta)}$are defined respectively by:

$$\left({}_xP_m^{(\alpha_1,\beta_1)}f\right)(x,y)=\sum\nolimits_{k=0}^{m}p_{mk}(x)f\left(\frac{k+\alpha_1}{m+\beta_1},y\right), \tag{2.1}$$

$$\left({}_yP_n^{(\alpha_2,\beta_2)}f\right)(x,y)=\sum\nolimits_{l=0}^{n}p_{nl}(y)f\left(x,\frac{l+\alpha_2}{n+\beta_2}\right). \tag{2.2}$$

It is easy to see that ${}_xP_m^{(\alpha_1,\beta_1)}$and ${}_yP_n^{(\alpha_2,\beta_2)}$are linear and positive operators, well defined on $C_b(I^2)$.

Let $L_{m,n}: C_b(I^2)\to C_b(I^2)$ be the tensorial product of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.

$$L_{m,n}={}_xP_{my}^{(\alpha_1,\beta_1)}\circ P_n^{(\alpha_2,\beta_2)}. \tag{2.3}$$

Then, $L_{m,n}: C_b(I^2)\to C_b(I^2)$ associates to any $f\in C_b(I^2)$ the bivariate polynomial

$$L_{m,n}f(x,y)=\sum\nolimits_{k=0}^{m}\sum\nolimits_{l=0}^{n}p_{mk}(x)p_{n,l}(y)f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{l+\alpha_2}{n+\beta_2}\right) \tag{2.4}$$

It is well known (see for example [4] or [10]) that the operator (2.4) has the following properties:

Lemma 2.1. If $e_{ij}:I^2\to\mathbb{R}\ (i,j\in\mathbb{N},0\le i+j\le 2)$ are the test functions the following equalities hold

(i) $(L_{m,n}e_{00})(x,y)=1$;

(ii) $(L_{m,n}e_{10})(x,y)=x+\frac{\alpha_1-\beta_1 x}{m+\beta_1}$;

(iii) $(L_{m,n}e_{01})(x,y)=y+\frac{\alpha_2-\beta_2 y}{n+\beta_2}$;

(iv) $(L_{m,n}e_{20})(x,y)=x^2+\frac{mx(1-x)+(\alpha_1-\beta_1x)(2mx+\beta_1x+\alpha_1)}{(m+\beta_1)^2}$;

(v) $(L_{m,n}e_{02})(x,y)=y^2+\frac{ny(1-y)+(\alpha_2-\beta_2y)(2ny+\beta_2y+\alpha_2)}{(m+\beta_2)^2}$;

for any $(x,y)\in I^2$.

Lemma 2.2 *The operator (2.4) is linear and positive.*

Definition 2.1. Let $S_{m,n}:C_b(I^2)\to C_b(I^2)$ be the boolean sum of ${}_xP_m^{(\alpha_1,\beta_1)}$ and ${}_yP_n^{(\alpha_2,\beta_2)}$, i.e.

$$S_{m,n}={}_xP_m^{(\alpha_1,\beta_1)}+{}_yP_n^{(\alpha_2,\beta_2)}-{}_xP_m^{(\alpha_1,\beta_1)}\circ{}_yP_n^{(\alpha_2,\beta_2)} \tag{2.5}$$

The operator $S_{m,n}$ will be called GBS operator of Stancu type.

By direct computation, one obtains:

Lemma 2.3. If $S_{m,n}:C_b(I^2)\to C_b(I^2)$ is the GBS operator of Stancu type, then

$$\begin{aligned}&\left(S_{m,n}f\right)(x,y)=\\&\sum\nolimits_{k=0}^{m}\sum\nolimits_{l=0}^{n}p_{mk}(x)\,p_{nl}(y)\times\left\{f\left(\tfrac{k+\alpha_1}{m+\beta_1},y\right)\right.\\&\left.+f\left(x,\tfrac{l+\alpha_2}{n+\beta_2},y\right)-f\left(\tfrac{k+\alpha_1}{m+\beta_1},\tfrac{l+\alpha_2}{n+\beta_2}\right)\right\}\end{aligned} \tag{2.6}$$

for any $f\in C_b(I^2)$ and any $(x,y)\in I^2$.

Remark 2.1. For $\alpha_1=\beta_1=\alpha_2=\beta_2=0$, the GBS operator of Stancu type is reduced to the GBS operator of Bernstein type, which interpolates any function $f\in C_b(I^2)$ on the boundary of the unit square I^2. If $\alpha_1=\beta_1=0$ and $\alpha_2\neq 0,\beta_2\neq 0$, the corresponding operator interpolates any $f\in C_b(I^2)$ on the left and respectively on the right side of the boundary of unit square I^2. Others particular cases of the GBS operator of Stancu type can be discussed in a similar way.

Theorem 2.1. *For any $f \in C_b(I^2)$,the sequence $\{S_{m,n}f\}_{m,n\in\mathbb{N}}$ converges to f, uniformly on I^2 as m and n tend to infinity*

Proof. Let us to introduce the following notations

$$u_m(x) = \frac{\alpha_1 - \beta_1 x}{m + \beta_1},$$
$$v_n(y) = \frac{\alpha_2 - \beta_2 y}{n + \beta_2},$$
$$w_m, n(x,y) = x^2 + y^2 + \frac{mx(1-x) + (\alpha_1 - \beta_1 x)(2mx + \beta_1 + \alpha_1)}{(m+\beta_1)^2}$$
$$+ \frac{ny(1-y) + (\alpha_2 - \beta_2 y)(2ny + \beta_2 + \alpha_2)}{(n+\beta_2)^2}.$$

Then the results contained in Lemma 2.1 can be written in the form

$(L_{m,n}e_{00})(x,y) = 1;$

$(L_{m,n}e_{10})(x,y) = x + u_m(x);$

$(L_{m,n}e_{01})(x,y) = y + v_n(y);$

$(L_{m,n}(e_{20} + e_{02}))(x,y) = x^2 + y^2 + w_{m,n}(x,y)$, for any $(x,y) \in I^2$.

Because the sequences $\{u_m(x)\}_{m\in\mathbb{N}}$, $\{v_n(x)\}_{n\in\mathbb{N}}$ and $\{w_{m,n}(x)\}_{m,n\in\mathbb{N}}$ tend to zero, uniformly on I^2 as m and n tend to infinity, we can apply the Korovkin - type theorem for the approximation of B-continuous functions due C.Badea, I.Badea and H.H.Gonska (see [2]. Applying this theorem, it follows that $S_{m,n}f$ tend to f, uniformly on I^2, for any $f \in C_b(I^2)$ as m and n tend to infinity.

Next the approximation order of any function $f \in C_b(I^2)$ by $S_{m,n}f$ will be established, using the mixed modulus of smoothness (see [3]). We need the following result, due to C. Badea and C. Cottin [see [3]).

Theorem 2.2. *Let X and Y be compact real intervals. Furthermore, let $L : C_b(X,Y) \to C_b(X,Y)$ be a positive linear operator and U the associated GBS operator. Then, for all $f \in C_b(X,Y)$, $(x,y) \in X \times Y$ and $\delta_1,\delta_2 > 0$ the inequality*

$$\begin{aligned}&|(f - Uf)(x,y)| \le |f(x,y)| \cdot |1 - L(x;x,y| + \\ &\{L(1;x,y) + \tfrac{1}{\delta_1}\sqrt{L((x-\circ)^2;x,y)} + + \tfrac{1}{\delta_2}\sqrt{L((y-*)^2;x,y)} + \\ &+ \tfrac{1}{\delta_1\delta_2}\sqrt{L((x-\circ)^2(y-*)^2;x,y)}\,\}\,\omega_{mixed}(\delta_1,\delta_2)\end{aligned} \tag{2.7}$$

holds.

Lemma 2.4. The bivariate operator of Stancu verifies the following equalities:

(i) $L_{m,n}((x-\circ)^2;x,y)=\frac{mx(1-x)+(\alpha_1-\beta_1 x)^2}{(m+\beta_1)^2}$;
(ii) $L_{m,n}((y-*)^2;x,y)=\frac{ny(1-y)+(\alpha_2-\beta_2 y)^2}{(n+\beta_2)^2}$;
(iii) $L_{m,n}((x-\circ)^2(y-*)^2=\frac{1}{(m+\beta_1)^2(n+\beta_2)^2}\times$
$\left\{mx(1-x)+(\alpha_1-\beta_1 x)^2\right\}\times\left\{ny(1-y)+(\alpha_2-\beta_2 y)^2\right\}.$

Proof. The equalities follow from the linearity of L_{mn} and Lemma 2.1. □

Theorem 2.3. *The GBS operators of Stancu S_{mn} verify the inequality:*

$$\begin{aligned}&|S_{m,n}f(x,y)-f(x,y)|\leq\\&\left\{\tfrac{1}{\delta 1}\cdot\tfrac{1}{m+\beta_1}\sqrt{\tfrac{m}{4}+(\alpha_1-\beta_1 x)^2}+\tfrac{1}{\delta_2}\sqrt{\tfrac{n}{4}+(\alpha_2-\beta_2 y)^2}+\right.\\&\left.+\tfrac{1}{\delta_1\delta_2}\cdot\tfrac{1}{(m+\beta_1)(n+\beta_2)}\sqrt{\{\tfrac{m}{4}+(\alpha_1-\beta_1 x)^2\}\{\tfrac{n}{4}+(\alpha_2-\beta_2 y)^2\}}\right\}\times\\&\times\omega_{mixed}(\delta_1\delta_2),\end{aligned}\tag{2.8}$$

f*or any* $\delta_1,\delta_2>0$ *and any* $(x,y)\in I^2$.

Proof. We apply the Lemma 2.4 and the inequalities $x(1-x)\leq\frac{1}{4},\ y(1-y)\leq\frac{1}{4}$ for any$(x,y)\in I^2$.□

Remark 2.2. The inequality (2.8) give us the order of the local approximation of f by $S_{m,n}f$.

The order of the global approximation of $f\in C_b(I^2)$ by $S_{m,n}f$ is expressed in

Theorem 2.4. The GBS operator of Stancu verify the following inequality:

$$|S_{m,n}f(x,y)-f(x,y)|\leq\frac{9}{4}\omega_{mixed}\left(\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1},\frac{\sqrt{n+4\alpha_2^2}}{n+\beta_2}\right)\tag{2.9}$$

Proof. Taking into account that $(\alpha_1-\beta_1 x)^2\leq\alpha_1^2$ and $(\alpha_2-\beta_2 y)^2\leq\alpha_1^2$ for any $(x,y)\in I^2$, from Theorem 2.3, we get:

$$\begin{aligned}&|S_{m,n}f(x,y)-f(x,y)|\leq\\&\left\{\frac{1}{2\delta_1}\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}+\frac{1}{2\delta_2}\frac{\sqrt{n+4\alpha_2^2}}{n+\beta_2}+\frac{\sqrt{(m+4\alpha_1^2)(n+4\alpha_2^2)}}{4\delta_1\delta_2(m+\beta_1)(m+\beta_2)}\right\}\omega_{mixed}(\delta_1\delta_2).\end{aligned}$$

Choosing then

$$\delta_1 = \frac{\sqrt{m + 4\alpha_1^2}}{m + \beta_1}; \qquad \delta_2 = \frac{\sqrt{n + 4\alpha_2^2}}{n + \beta_2};$$

it follows (2.9) and the proof ends □.

Remark 2.3. The inequality (2.9) can be more rafinated, taking into account of the values of α_1,α_2 with respect β_1 and β_2.

References

[1] Altomare, F., Campiti, M., *Korovkin -type Approximation Theory and its Application*, de Gruyter Series Studies in Mathematics, Vol.17, Walter de Gruyter &Co, Berlin, New-York, 1994.

[2] Badea, C., Badea, I., Gonska, H.H., *A test function theorem and approximation by pseudopolynomials*, Bull. Austral. Math. Soc. 34(1986), 53-64.

[3] Badea, C., Cottin., *Korovkin-type theorems for Generalided Boolean Sum Operators*, Colloquia Mathematica Societatis Janos Bolyai, 58, Approximation Theory, Kecskemet(Hungary)(1990), 51-68.

[4] Barbosu, D., *A Voronovskaia-type theorem for the bivariate operator of Stancu* (submitted).

[5] Bőgel, K., *Mehrdimensionale Differentiation von Funktionen mehrer Veränderlicher*, J. Reine Angew. Math.170(1934), 197-217.

[6] Bőgel, K., *Ubër die mehrdimensionale Differentiation Integration und beschrankte Variation*, J. Reine Angew. Math.173(1935), 5-29.

[7] Dobrescu, E., Matei, I., *Aproximarea prin polinoame de tip Bernstein a funcţiilor bidimensional continue*, Anal. Univ. Timişoara, Seria Ştiinţe matematice fizice, Vol. IV(1966), 85-90.

[8] Stancu, D.D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Universitatis Babes-Bolyai, 14(1969), 31-45(in Romanian).

[9] Stancu, D. D., Coman, Gh., Agratini, O., Trâmbiţaş, R., *Analiză numerică şi teoria aproximării*, Vol. I, Presa Universitară Clujeană, 2001.

[10] Stancu, F., *Aproximarea funcţiilor de două şi mai multe variabile cu ajutorul operatorilor liniari pozitivi*, Ph. D. Thesis, Cluj-Napoca (1984) (in Romanian).

Department of Mathematics and Computer Science,
North University of Baia Mare, Victoriei 76, 4800 Baia Mare, Romania
E-mail address: dbarbosu@univer.ubm.ro