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ON THE ORTHOGONAL POLYNOMIALS OF PARETO AND APPLICATIONS ON MATHEMATICAL MODELS

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Dedicated to Professor D.D. Stancu on his 75^{th} *birthday*

Abstract. The advantage of determining the differential stochastic equations with applications in Economy, through an orthogonal polynomial system unfolds from the fact that we can approximate the solution, improve the solution, and determine the degree of precision of the approximation.

We shall make such a determination through the orthogonal polynomials associated with the law of Pareto.

The advantage of determining the differential stochastic equations with applications in Economy, through an orthogonal polynomial system unfolds from the fact that we can approximate the solution, improve the solution, and determine the degree of precision of the approximation.

We shall make such a determination through the orthogonal polynomials associated with the law of Pareto. By these means we will step into Numerical Analysis.

I) We have seen the stochastic equations applied in Economy such that (see [4])

$$\frac{dS}{S} = \mu dt + \sigma dX \tag{1.1}$$

or the Black & Scholem variante, that is to say:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1.2}$$

where r and σ are constants -if we have a simplyfied version of the problem - but in a more general case of the problem, these constants are dependent on t.

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This solution may be expressed by orthogonal polynomials, that is to say we found the formula:

$$V(x,\tau)=V(x,0)\exp(A,\tau)=\sum_{n=0}^{\infty}a_n(\tau)P_n(x) \tag{1.3}$$

where $P_n(x)$ for $n=0,1,2,3...$ are the orthogonal polynomial associated for the Pareto distribution, and $a_n(\tau)$ are the coefficients (see [4]).

To obtain (1.3), we have the following hypothesis: $\frac{r(t)}{\sigma^2(t)}=c$ and $\alpha=c-3>0$, then the found Pareto law had the following probability density:

$$f(x-\lambda)=\frac{\alpha\lambda^{\alpha}}{x^{\alpha+1}} \quad \text{where} \quad x>\lambda \quad \text{and} \quad \lambda=1. \tag{1.4}$$

In order to construct the orthogonal polynomial system associated to the weights $\rho(x)=f(x-1)$ given by (1.4) - we must use the following formula, (see [3])

$$P_{n(x)}=b_n\begin{vmatrix} M_0 & M_1 & .... & M_n \\ M_1 & M_2 & .... & M_{n+1} \\ .... & .... & .... & .... \\ 1 & x & .... & x^n \end{vmatrix} \tag{1.5}$$

where $M_n=\int\limits_1^{\infty}x^n\frac{\alpha}{x^{\alpha+1}}dx$ and $b_n=$ the normalization constant.

M_n is the moment of order n of the density $\rho(x)=f(x-1)=\frac{\alpha}{x^{\alpha+1}}$ for $x>1$ as $\lambda=1$

The M_n moments for $n=0,1,2,3...$are estimated by improper integrals as the superior limit is ∞, which exist if $\alpha>n$.

Clearly, by the following transformation: $u=\frac{1}{x}$ in this integral $\int\limits_1^{\infty}\alpha x^{n-\alpha-1}dx$ we have the results

$\int\limits_0^1\alpha u^{-n+\alpha-1}du=\frac{\alpha}{\alpha-n}$ if $\alpha>n$ as if $\alpha<1$ or if $\alpha=1$ the integral does not exist. The results are: $M_n=\frac{\alpha}{\alpha-n}$ for $\alpha>n$ that is to say: $M_0=1$, $M_1=\frac{\alpha}{\alpha-1},...,M_k=\frac{\alpha}{\alpha-k},...,M_n=\frac{\alpha}{\alpha-n}$.

The orthogonal polynomials associated with the Pareto law, that is to say $P_n(x)$ given by (1.5) are expressed by the following relation:

$$P_0=b_0, \quad P_1=b_1\left[x-\frac{\alpha}{\alpha-1}\right],$$

$$P_2 = b_2 \begin{bmatrix} x^2 \frac{\alpha}{(\alpha-1)^2(\alpha-2)} + x\frac{\alpha}{(\alpha-1)(\alpha-2)(\alpha-3)} + \\ + \frac{\alpha^2(2\alpha+1)}{(\alpha-1)(\alpha-3)(a^2-2)^2} \end{bmatrix}, \ldots$$

$$P_n(x) = b_n [x^n........] \tag{1.6}$$

for $\alpha > n$.

(II)

1) In applications we may conclude early that formula (1.3) which renders the value V of the option will be trucated by the orthogonal polynomials of Pareto, having the condition $\alpha > n$ for the existence of these polynomials.

In order to know in which manner the above mentioned condition is implied in the applications, we must revise the definition of $\alpha = c - 3 = \frac{r(t)}{\sigma^2(t)} - 3 > 0$.

For the $r(t)$ interest rate of the active S, which is the stochastic processes in the capital market, we may say there are few mentioned models(see [1]).

Also, for the $\sigma^2(t)$ variance, or the $\sigma(t)$ volatility of active S, there are several mathematical models, inclusively by Hermite and Laguerre orthogonal polynomials(see [2]).

Therefore in a given time period $t \in (a, b)$, the strategy over the $r(t)$ rate and its relation with $\sigma^2(t)$, will be optimized. By these means, the value of α will be determined knowingly, through knowledge of mathematical models, and then we will see how the (1,3) formula will be truncated.

2) Another application of Pareto's law will appear from the following proposition:

a) *"If x obeys to the Pareto's law with α and λ, then the variable* $y = \log(1 + \frac{x}{\lambda})$ *will obey to the exponential law of* $\frac{1}{\alpha}$*mean.*

Therefrom the following statement

b)*"If $x_{(1)}, x_{(2)}, ...x_{(n)}$ is a sample of ordinated values of a distribution x of Pareto, then the variables $x_{(k)}$ will be expressed such as:*

$$x_{(k)} = \lambda \left[\prod_{j=1}^{k} (1 + v_j) - 1 \right]$$

for $k = 1, 2...n$; where v_j are independent Pareto variables".

Certainly, the b) statement results from a) especially due to variable y obeying to an exponential law and we apply the exponential law theorems concerning the ordinated variables extracted from an n volume sample.

In conclusion, the independent Pareto variables will be used in order statistics.

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