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ON APPROXIMATION PROPERTIES OF STANCU'S OPERATORS

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Dedicated to Professor D.D. Stancu on his 75^{th} birthday

Abstract. The purpose of the paper is to present pointwise and uniform approximation theorems for some Stancu's operators using the classical moduli of smoothness and the second modulus of smoothness of Ditzian - Totik.

1. **Introduction**

One of the most studied operator (see e.g. the bibliography of [1]) is $B_n^\alpha : C[0,1] \to C[0,1]$,

$$B_n^\alpha(f,x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha) \cdot f\left(\frac{k}{n}\right), \quad n = 1,2,\dots, \quad x \in [0,1], \quad \alpha \geq 0, \tag{1}$$

where

$$w_{n,k}(x,\alpha) = \begin{pmatrix} n \\ k \end{pmatrix} \cdot \frac{\prod_{i=0}^{k-1} (x+i\alpha) \prod_{j=0}^{n-k-1} (1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)} \tag{2}$$

and α is a parameter which may depend only on the natural number n. This positive linear polynomial operator was introduced by D. D. Stancu in [15]. In the case $\alpha = 0$, B_n^α is the Bernstein operator B_n given by

$$B_n(f,x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} \cdot f\left(\frac{k}{n}\right). \tag{3}$$

The Stancu - Kantorovich polynomial operator was defined in [14] as follows: $K_n^\alpha : L^p[0,1] \to L^p[0,1], \quad 1 \leq p \leq \infty$,

$$K_n^\alpha(f,x) = (n+1) \sum_{k=0}^{n} w_{n,k}(x,\alpha) \cdot \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du, \quad n = 1,2,\dots, \quad x \in [0,1] \tag{4}$$

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and α and $w_{n,k}(x,\alpha)$ have the same meaning as above. For $\alpha = 0$, K_n^α is the Kantorovich operator K_n given by

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du. \tag{5}$$

The spaces $L^p[0,1]$, $1 \le p \le \infty$, are endowed with the norm

$$\|f\|_p = \left\{\int_0^1 |f(x)|^p\, dx\right\}^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ we consider $C[0,1]$ instead of $L^\infty[0,1]$ with

$$\|f\| = \|f\|_\infty = \sup\left\{|f(x)| \ : \ x \in [0,1]\right\}.$$

The corresponding operator to Bernstein operator on the positive semiaxis is the so - called Szász - Mirakjan operator defined by $S_n : C_B[0,\infty) \to C_B[0,\infty)$,

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\cdot f\left(\frac{k}{n}\right), \quad n = 1,2,\dots, x \in [0,\infty), \tag{6}$$

where $C_B[0,\infty)$ denotes the set of all bounded and continuous functions on $[0,\infty)$ endowed with the norm

$$\|f\|_* = \sup\left\{|f(x)| \ : \ x \in [0,\infty)\right\}.$$

The operator S_n was generalized by Stancu in [16], obtaining S_n^β operators

$$S_n^\beta(f,x) = (1+n\beta)^{-x/\beta}\cdot\sum_{k=0}^{\infty}\left(\beta+\frac{1}{n}\right)^{-k}\cdot\frac{x(x+\beta)\dots(x+(k-1)\beta)}{k!}\cdot f\left(\frac{k}{n}\right), \tag{7}$$

where $\beta > 0$ is a parameter depending on the natural number n.

Furthermore, in the paper [17], Stancu has introduced a generalization of the well - known Baskakov operator $V_n : C_B[0,\infty) \to C_B[0,\infty)$,

$$V_n(f,x) = \sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-n-k}\cdot f\left(\frac{k}{n}\right), n = 1,2,\dots, x \in [0,\infty), \tag{8}$$

defined by

$$V_n^\gamma(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x,\gamma)\cdot f\left(\frac{k}{n}\right), \tag{9}$$

where

$$v_{n,k}(x,\gamma) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}\cdot\frac{\prod_{i=0}^{k-1}(x+i\gamma)\ \prod_{j=0}^{n-1}(1+j\gamma)}{\prod_{r=0}^{n+k-1}(1+x+r\gamma)} \tag{10}$$

and $\gamma \geq 0$ depends on the natural number n.

The purpose of this paper is to establish pointwise and uniform approximation properties for the operators $(1)-(2)$, (4), (7) and $(9)-(10)$. On the other hand the paper will be a survey of some results given by the author regarding the above mentioned Stancu's operators.

To establish these results we shall use the following notations:

$$\omega(g,t)_p = \sup_{0<h\leq t}\left\{\int_0^1 |g(x+h)-g(x)|^p\,dx\right\}^{1/p},$$
$$g\in L^p[0,1],\quad 1\leq p<\infty,\quad x,x+h\in[0,1];$$

$$\omega_2(g,t) = \sup_{0<h\leq t}\ \sup_{x,x\pm h\in I} |g(x+h)-2g(x)+g(x-h)|,$$
$$g\in C(I),\ I=[0,1]\ \text{ or }\ I=[0,\infty);$$

$$\omega_2^{\varphi}(g,t) = \sup_{0<h\leq t}\ \sup_{x\pm h\varphi(x)\in I} |g(x+h\varphi(x))-2g(x)+g(x-h\varphi(x))|,$$
$$g\in C[0,1] \quad \text{and}\ \ \varphi(x)=\sqrt{x(1-x)},$$
$$g\in C_B[0,\infty) \quad \text{and}\ \ \varphi(x)=\sqrt{x}\ \ \text{or}$$
$$g\in C_B[0,\infty) \quad \text{and}\ \ \varphi(x)=\sqrt{x(1+x)};$$

$$\omega_2^{\varphi}(g,t)_p = \sup_{0<h\leq t}\left\{\int_0^1 |g(x+h\varphi(x))-2g(x)+g(x-h\varphi(x))|^p\,dx\right\}^{1/p},$$
$$g\in L^p[0,1],\ 1\leq p<\infty,\ x\pm h\varphi(x)\in[0,1]$$
$$\text{and}\ \ \varphi(x)=\sqrt{x(1-x)},\ x\in[0,1];$$

$$\omega_2^{\phi}(g,t) = \sup_{0<h\leq t}\ \sup_{x\pm h\phi(x)\in[0,\infty)} |g(x+h\phi(x))-2g(x)+g(x-h\phi(x))|,$$
$g\in C_B[0,\infty)$ and $\phi:[0,\infty)\to\Re$ is an admissible step - weight function (see [3]).

Here we mention that throughout this paper C and C_0 denote absolute constants and not necessarily the same at each occurrence.

2. Theorems

In [5, Theorem 1] we have proved the following

Theorem 1. *For* $f \in C[0,1]$ *and* $x \in [0,1]$ *we have*

$$|B_n^\alpha(f,x) - f(x)| \;\le\; C\;\omega_2\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)}\cdot x(1-x)}\;\right).$$

Remark 1. We can obtain the estimate of Theorem 1 with $C = 2$ using [13, p. 255, Theorem 2.1].

Furthermore, by [6, p. 100, Theorem 1], we have

Theorem 2. *Let* $f \in C[0,1]$ *and* $\alpha = \alpha(n) = o(n^{-1})$, $\alpha\; n \le 1$, $n = 1, 2, \ldots$. *Then*

$$|B_n^\alpha(f,x) - f(x)| \;\le\; C \cdot \frac{x(1-x)}{n}, \quad x \in [0,1], \quad n = 1, 2, \ldots$$

holds exactly when $\omega_2\,(f,h) \;\le\; C\;h^2,\;\; h > 0.$

Using [2, p. 79, Theorem A] or [6, p. 100, Theorem 3], we get

Theorem 3. *For* $f \in C[0,1]$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *we have*

$$\|B_n^\alpha(f) - f\| \;\le\; C\;\omega_2^\varphi\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)}}\right).$$

The next result requires the following lemma (see [12, p. 317, (2.1)] or [19]):

Lemma 1. *Let* $f \in C[0,1]$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. *Then*

$$\frac{1}{n}\;\|\varphi^2(B_n(f))''\| \;\le\; C_0\;\|B_n(f) - f\|,$$

where C_0 *is an absolute constant.*

Then our result is (see [7, p. 2, Theorem 3]) :

Theorem 4. *Let* $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\alpha = \alpha(n)$, $2\;C_0\;\alpha\;n \le 1$, $n = 1, 2, \ldots$, *where* C_0 *denotes the absolute constant of Lemma 1 above. Then there exists an absolute constant* $C > 0$ *such that*

$$C^{-1}\;\|B_n(f) - f\| \;\le\; \|B_n^\alpha(f) - f\| \;\le\; C\;\|B_n(f) - f\|$$

and

$$C^{-1}\;\omega_2^\varphi(f, n^{-1/2}) \;\le\; \|B_n^\alpha(f) - f\| \;\le\; C\;\omega_2^\varphi(f, n^{-1/2}).$$

Hence, in view of [3, p. 177, (9.3.3)], we obtain immediately

Corollary 1. *Let* $f \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, $\alpha = \alpha(n)$ *with* $2\ C_0\ \alpha\ n \le 1$, $n = 1, 2, \ldots$ *and* $0 < \delta < 2$. *Then*

$$\|B_n^\alpha(f) - f\| \ = \ O(n^{-\delta/2}) \quad \textit{iff} \quad \omega_2^\varphi\,(f,h) \ = \ O(h^\delta), \quad h > 0.$$

The following results will be in connection with the operator K_n^α. More precisely, we have (see [8, Theorem 1, Lemma 2 and Theorem 3]) :

Theorem 5. *Let* $f \in L^p[0,1]$, $1 \le p \le \infty$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. *Then there exists* $C > 0$ *such that*

(*i*) $\|K_n^\alpha(f) - f\|_p \ \le \ C \ \left\{\omega_2^\varphi(f, n^{-1/2})_p + n^{-1}\ \|f\|_p\right\}$,
where $\alpha = \alpha(n) = O(n^{-1})$ *and* $1 < p \le \infty$;

(*ii*) $\|K_n^\alpha(f) - f\|_1 \ \le \ C \ \left\{\omega_2^\varphi(f, n^{-1/2})_1 + n^{-1}\ \|f\|_1\right\}$,
where $\alpha = \alpha(n) = O(n^{-4})$.

For the converse result we need a lemma :

Lemma 2. *For* $f \in L^p[0,1]$, $1 < p \le \infty$ *and* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *we have*

$$\frac{1}{n}\ \|\varphi^2(K_n(f))''\|_p \ \le \ C_0\ \|K_n(f) - f\|_p,$$

where C_0 *is an absolute constant.*

Remark 2. The above Lemma does not hold for $p = 1$ (see [8, Remark 2]).

Our result is

Theorem 6. *Let* $f \in L^p[0,1]$, $1 < p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\alpha = \alpha(n)$, $p/(p-1)\ C_0\ \alpha\ n \le \delta < 1$, $n = 1, 2, \ldots$, *where* C_0 *denotes the absolute constant of Lemma 2. Then*

$$(1-\delta)\ \|K_n(f) - f\|_p \ \le \ \|K_n^\alpha(f) - f\|_p \ \le \ (1+\delta)\ \|K_n(f) - f\|_p$$

and there exists an absolute constant $C > 0$ *such that*

$$C^{-1}\left[\omega_2^\varphi(f, n^{-1/2})_p + \omega(f, n^{-1})_p\right] \le \|K_n^\alpha(f) - f\|_p \le C\left[\omega_2^\varphi(f, n^{-1/2})_p + \omega(f, n^{-1})_p\right].$$

In what follows we give the theorems concerning to the operator S_n^β using [9, p. 62, Theorem 1] and [10] :

Theorem 7. *For $f \in C[0,\infty)$ and $x \in [0,\infty)$ we have*

$$|S_n^\beta(f,x) - f(x)| \ \le \ 2\ \omega_2\left(f, \sqrt{\left(\beta + \frac{1}{n}\right)\frac{x}{2}}\right).$$

Theorem 8. *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$. Then*

$$\|S_n^\beta(f) - f\|_* \ \le \ C\ \omega_2^\varphi\left(f, \sqrt{\frac{1}{n} + \beta}\right).$$

Theorem 9. *Let $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$ and $\beta = \beta(n)$, $2\ C_0\ \beta\ n \le \delta < 1$, $n = 1,2,\ldots,$ where C_0 denotes the absolute constant of Lemma 3 below. Then*

$$(1-\delta)\ \|S_n(f) - f\|_* \ \le \ \|S_n^\beta(f) - f\|_* \ \le \ (1+\delta)\ \|S_n(f) - f\|_*$$

and there exists an absolute constant $C > 0$ such that

$$C^{-1}\ \omega_2^\varphi(f, n^{-1/2}) \ \le \ \|S_n^\beta(f) - f\|_* \ \le \ C\ \omega_2^\varphi(f, n^{-1/2}).$$

Lemma 3. [19] *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x}$, $x \in [0,\infty)$. Then*

$$\frac{1}{n}\ \|\varphi^2(S_n(f))''\|_* \ \le \ C_0\ \|S_n(f) - f\|_*,$$

where C_0 is an absolute constant.

Finally, we give the results about the operator V_n^γ. This operator is linear, positive and bounded, but it does not preserve the linear functions. Therefore we consider the following two cases :

a)

$$L_n^\gamma(f,x) \ = \ a_0(n) \cdot V_{n_0}^\gamma(f,x) \ + \ a_1(n) \cdot V_{n_1}^\gamma(f,x), \tag{11}$$

where

$$n = n_0 < n_1 \le A\ n, \qquad |a_0(n)| + |a_1(n)| \ \le \ A,$$
$$a_0(n) + a_1(n) = 1, \qquad a_0(n) \cdot n_0^{-1} + a_1(n) \cdot n_1^{-1} = 0$$

and $\gamma = \gamma(n) \leq B/(4n)$, $n = 1, 2, \ldots$, $0 < B < 1$. Here A and B are given absolute constants. Following [11] (see also [4]), we have

Theorem 10. *Let $L_n^\gamma : C_B[0,\infty) \to C_B[0,\infty)$ be given by (11), $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $\phi : [0,\infty) \to \Re$ be an admissible step - weight function of the Ditzian - Totik modulus and $\gamma = \gamma(n) \leq B/(4n)$, $n = 1, 2, \ldots$, $0 < B < 1$. Then*

$$|L_n^\beta(f,x) - f(x)| \;\leq\; C\ \omega_2^\phi \left(f, n^{-1/2} \cdot \frac{\varphi(x)}{\phi(x)} \right), \quad x \in [0,\infty).$$

In particular, we obtain a local estimation of the approximation error for $\phi = 1$:

$$|L_n^\beta(f,x) - f(x)| \;\leq\; C\ \omega_2^\phi \left(f, \sqrt{\frac{x(1+x)}{n}} \right)$$

and we get a uniform (global) estimation of the approximation error for $\phi = \varphi$:

$$\|L_n^\beta(f) - f\|_* \;\leq\; C\ \omega_2^\varphi (f, n^{-1/2}).$$

b)

$$\tilde{V}_n^\gamma(f,x) = \sum_{k=0}^{\infty} \tilde{v}_{n,k}(x,\gamma) \cdot f\left(\frac{k}{n}\right), \tag{12}$$

where

$$\tilde{v}_{n,k}(x,\gamma) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \cdot \frac{\prod_{i=0}^{k-1}\ (x+i\gamma) \cdot \prod_{j=1}^{n}\ (1+j\gamma)}{\prod_{r=1}^{n+k}\ (1+x+r\gamma)} \tag{13}$$

(see also [18]). By [10], we have

Theorem 11. *For $\tilde{V}_n^\gamma : C_B[0,\infty) \to C_B[0,\infty)$ given by (12) - (13), $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $0 < \gamma < 1$ we have*

$$\|\tilde{V}_n^\gamma(f) - f\|_* \;\leq\; C\ \omega_2^\varphi \left(f, \sqrt{\frac{1}{n} + \frac{\gamma}{1-\gamma}} \right).$$

Theorem 12. *Let $f \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$ and $\gamma = \gamma(n)$, $2\ C_0 \cdot (\gamma/(1-\gamma)) \cdot n \leq \delta < 1$, $n = 1, 2, \ldots,$ where C_0 denotes the absolute constant of Lemma 4 below. Then*

$$(1-\delta)\ \|V_n(f) - f\|_* \;\leq\; \|\tilde{V}_n^\gamma(f) - f\|_* \;\leq\; (1+\delta)\ \|V_n(f) - f\|_*$$

and there exists an absolute constant $C > 0$ such that

$$C^{-1}\ \omega_2^{\varphi}(f, n^{-1/2})\ \le\ \|\tilde{V}_n^{\gamma}(f) - f\|_*\ \le\ C\ \omega_2^{\varphi}(f, n^{-1/2}).$$

Lemma 4. [19] *Let $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$. Then*

$$\frac{1}{n}\ \|\varphi^2(V_n(f))''\|_*\ \le\ C_0\ \|V_n(f) - f\|_*,$$

where C_0 is an absolute constant.

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