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TWO-VARIABLE VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper we guarantee the solution for two-variable variational-hemivariational inequalities and we give some applications.

1. Introduction

The aim of this paper is to establish a two-variable result concerning the hemivariational inequalities. These inequalities appear as a generalisation of variational inequalitis, but they are more general than these ones, having applications in several branches of mathematics, mechanics, economy engineering.

The paper is organized as follows. In the Section 2 we formulate the problem and give some notions and results which will be used later. In Section 3 we establish the main results of this paper, i.e. we guarantee solution for hemivariational inequality. Finally in Section 4 we give some applications. More precisely, we obtain a Brouwer's type variational inequality, the Schauder fixed point theorem (and Brouwer fixed point theorem), a hemivariational inequality of Panagiotopoulos-Fundo-Rădulescu type, and a result concerning the Nash equilibrium theory.

2. Preliminaries

Let X be a Banach space, X^* its dual. We consider the following hypotheses:

(H_T) $T : X \to L^p(\Omega, \mathbb{R}^k)$ is a linear, continuous operator, where $p \in [1, \infty)$, $k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$.

(H_j) $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R})$ and $h_2 \in L^\infty(\Omega, \mathbb{R})$

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such that

$$|w| \leq h_1(x) + h_2(x)|y|^{p-1}$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $w \in \partial j(x,y)$, where $\partial j(x,y)$ is the Clarke generalized gradient of j, see [4], i.e. $\partial j(x,y) = \{w \in \mathbb{R}^k : \langle w, z\rangle \leq j_y^0(x,y;z), \text{ for all } z \in \mathbb{R}^k\}$ where $j_y^0(x,y;z)$ is the partial Clarke derivative of the locally Lipschitz mapping $j(x,\cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $z \in \mathbb{R}^k$, where $x \in \Omega$, that is

$$j_y^0(x,y,z) = \operatorname{limsup}_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y'+tz) - j(x,y')}{t}.$$

Let K be a subset of X, $\mathcal{A} : K \times K \rightsquigarrow X^*$, $G : K \times X \rightsquigarrow \mathbb{R}$ two set-valued mappings with nonempty values. Under hypotheses (H_T) and (H_j) the main problem of this paper is the following

(P) Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + G(u, v-u) + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \subseteq \mathbb{R}_+.$$

Here $\sigma(\mathcal{A}(w,u), v-u) = \sup\limits_{x^* \in \mathcal{A}(w,u)} \langle x^*, v-u\rangle$. The (P) is equivalent with

(P') Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + \inf G(u, v-u) + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0.$$

The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle \cdot, \cdot \rangle$.

In order to state existence results for (P), we need some notions and preliminary results.

Definition 2.1. *Let K be convex.*

(i) *A set-valued mapping $\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous at $x_0 \in K$ *(udc at $x_0 \in K$) if for any $h \in X$, the real-valued function $x \mapsto \sigma(\mathcal{F}(x), h) = \sup\limits_{x^* \in \mathcal{F}(x)} \langle x^*, h\rangle$ is upper semicontinuous at x_0. $\mathcal{F}$ is upper demicontinuous on K (udc on K) if it is udc in every $x \in K$.*

(ii) *$\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous from the line segments in K *if the application $t \mapsto \sigma(\mathcal{F}(tx + (1-t)y), h)$ is upper semicontinuous on the interval* $[0,1]$, $\forall x, y \in K, h \in X$.

(iii) $F : K \to X^*$ *is said to be* w^*-demicontinuous in u_0 *if for any sequence* $\{u_n\} \subset K$ *converging to* u_0 *(in the strong topology), the image sequence* $\{F(u_n)\}$ *converges to* $F(u_0)$ *in the weak***-topology in* X^*.

Remark 2.1. (i) If $\mathcal{F}(x) = \{F(x)\}$, that is, if $\mathcal{F}$ is a single valued map, then $\mathcal{F}$ is udc at $u_0 \in K$ if and only if the operator $F : K \to X^*$ is w^*-demicontinuous at $u_0 \in K$.

(ii) If $\mathcal{F}(x) = \{F(x)\}$ is hemicontinuous, (see for example [8]), then $\mathcal{F}$ is udc from the line segments in K.

The $h \mapsto \sigma(\mathcal{F}(x), h)$ is a lower semicontinuous sublinear function.

Lemma 2.1. [11, Lemma 2.2] *Let* $\mathcal{F} : K \rightsquigarrow X^*$ *be an udc set-valued map with bounded values, i.e.* $\sup\limits_{x^* \in \mathcal{F}(x)} \|x^*\| < \infty$, $\forall x \in K$. *Then the function* $u \mapsto \sigma(\mathcal{F}(u), v-u)$ *is upper semicontinuous,* $\forall\ v \in K$.

Now, we recall some notions from [1]. Let Y, Z be two metric spaces and a set-valued map (with nonempty values) $F : Y \rightsquigarrow Z$.

Definition 2.2. *F is called* lower semicontinuous at $y \in Y$ *(lsc at y) if and only if for any $z \in F(y)$ and for any sequence $\{y_n\}$, converging to y, there exists a sequence $\{z_n\}$, $z_n \in F(y_n)$ converging to z.*

It is said to be lower semicontinuous (lsc) if it is lsc at every point $y \in Y$.

Let us consider a function $f : Graph(F) \to \mathbb{R}$. We define the *marginal function* $g : Y \to \mathbf{R} \cup \{+\infty\}$ by $g(y) = \sup_{z \in F(y)} f(y,z)$. We have the Maximum Theorem, see [1, Theorem 1.4.16, p.48].

Lemma 2.2. *If f and F are lower semicontinuous on Y, then the marginal function is also lower semicontinuous.*

Definition 2.3. *Let K be a convex subset of X and let Z be a topological vector space. The set-valued map $F : K \rightsquigarrow Z$ (with nonempty values) is* convex *if and only if* $\forall\ x_1, x_2 \in K,\ \forall\ \lambda \in [0,1] :\ \lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2)$.

Remark 2.2. $F : K \rightsquigarrow Z$ is convex if and only if $\forall\ x_i \in K,\ \forall\ \lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1, n \in \mathbb{N}^*,$ we have $\sum_{i=1}^n \lambda_i F(x_i) \subseteq F\left(\sum_{i=1}^n \lambda_i x_i\right).$

Definition 2.4. *The mapping* $F : K \subseteq X \rightsquigarrow X^*$ *is* monotone *if* $\langle f_1 - f_2, u - v\rangle \geq 0,\ \forall u, v \in K,\ \forall f_1 \in F(u), f_2 \in F(v)$.

Lemma 2.3.([12, Lemma 1.]) *If T and j satisfy the (H_T) and (H_j) respectively and $V_1,\ V_2$ are non-empty subsets of X, then the mapping defined by*

$$(u,v) \mapsto \int_\Omega j_y^0(x, Tu(x), Tv(x))dx,\ (u,v) \in V_1 \times V_2$$

is upper semicontinuous.

Lemma 2.4. [7] *Let X be a Hausdorff topological vector space, K a subset of X and for each $x \in K$, let $S(x)$ be a closed subset of X, such that*

(i) *there exists $x_0 \in K$ such that the set $S(x_0)$ is compact*;

(ii) *S is KKM-mapping, i.e. for each $x_1, x_2, \ldots, x_n \in K$, $co\{x_1, x_2, \ldots, x_n\} \subseteq \cup_{i=1}^n S(x_i)$, where co stands for the convex hull operator.*

Then $\bigcap\limits_{x \in K} S(x) \neq \emptyset$.

3. Main results on Existence of Solutions for (P)

We need some additional hypotheses to obtain solution for (P).

(H_G) (1) $G(u,0) \subseteq \mathbb{R}_+,\ \forall\ u \in K$;

(2) $G(u,\cdot)$ is convex, $\forall u \in K$;

(3) $G(\cdot,\cdot)$ is lsc on $K \times X$;

(4) $G(u,\cdot)$ is subhomogenous, i.e. $tG(u,y) \subseteq G(u,ty),\ \forall t \in [0,1],\ u \in K,\ y \in X$.

$(H_{\mathcal{A}})$ (1) $\mathcal{A}$ has bounded values, i.e. $\sup\limits_{x^* \in \mathcal{A}(u,v)} \|x^*\| < \infty,\ \forall u, v \in K$;

(2) $\mathcal{A}(v,\cdot) : K \rightsquigarrow X^*$ is udc on K, $\forall v \in K$;

(3) $\mathcal{A}(\cdot,u) : K \rightsquigarrow X^*$ is udc from the line segments in K, $\forall u \in K$.

(4) $\mathcal{A}(\cdot,u)$ has the monotonicity property

$$\sigma(\mathcal{A}(v,u), v-u) \geq \sigma(\mathcal{A}(u,u), v-u), \forall u, v \in K.$$

The main result of this paper is the following

Theorem 3.1. *Let K be a convex, closed subset of a Banach space X and $\mathcal{A} : K \times K \rightsquigarrow X^*$, $G : K \times X \rightsquigarrow \mathbb{R}$, $T : X \to L^p(\Omega, \mathbb{R}^k)$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfying $(H_{\mathcal{A}})$, (H_G), (H_T) and (H_j) respectively. In addition, if*

(H_{coer}) there exists a compact subset K_0 of K and $u_0 \in K$ such that

$$\{\sigma(\mathcal{A}(u,u), u_0 - u) + G(u, u_0 - u) + \int_\Omega j_y^0(x, Tu(x), Tu_0(x) - Tu(x))dx\} \cap \mathbb{R}_-^* \neq \emptyset,$$

for all $u \in K \setminus K_0$. Then (P) has at least a solution.

Proof. For $w \in K$, let

$$T_1(w) = \{u \in K : \sigma(\mathcal{A}(u,u), w-u) + \inf\ G(u, w-u) +$$
$$+ \int_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\};$$
$$T_2(w) = \{u \in K_0 : \sigma(\mathcal{A}(w,u), w-u) + \inf\ G(u, w-u) +$$
$$+ \int_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\}.$$

Step 1. $T_1(u_0) \subseteq K_0$, where u_0 is from (H_{coer}). Suppose that there exists $u \in T_1(u_0) \subset K$ such that $u \notin K_0$. from the definition of $T_1(u_0)$, we have that

$$\sigma(\mathcal{A}(u,u), u_0 - u) + \inf G(u, u_0 - u) + \int_\Omega j^0(x, Tu(x), Tu_0(x) - Tu(x))dx \geq 0.$$

But this contradicts the (H_{coer}). Therefore $T_1(u_0) \subseteq K_0$.

Step 2. We prove that $T_1 : K \rightsquigarrow K$ is KKM-mapping, i.e.

$$\forall w_1, ..., w_n \in K : co\{w_1, ..., w_n\} \subseteq \bigcup_{i=1}^n T_1(w_i).$$

Contrary, we suppose that there exist $\lambda_1, \ldots, \lambda_n \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ such that $\overline{w} = \sum_{i=1}^n \lambda_i w_i \notin T_1(w_i)$, for $i = \overline{1,n}$. Therefore

$$\sigma(\mathcal{A}(\overline{w}, \overline{w}), w_i - \overline{w}) + \inf\ G(\overline{w}, w_i - \overline{w}) +$$
$$+ \int_\Omega j^0(x, T\overline{w}(x), -Tw_i(x) - T\overline{w}(x))dx < 0,\ i = \overline{1,n}.$$

Let $\mathcal{I} = \{i = \overline{1,n} : \lambda_i \neq 0\}$. Multiplying the above inequalities by λ_i for $i \in \mathcal{I}$ and using the homogenity of T, we have

$$\sigma(\mathcal{A}(\overline{w}, \overline{w}), \lambda_i w_i - \lambda_i \overline{w}) + \lambda_i \inf\ G(\overline{w}, w_i - \overline{w}) +$$
$$+ \int_\Omega j^0(x, T\overline{w}(x), -T(\lambda_i w_i)(x) - T(\lambda_i \overline{w})(x))dx < 0, \forall i \in \mathcal{I}.$$

Adding the above relations for $i \in \mathcal{I}$ and using that $h \mapsto \sigma(\mathcal{A}(\overline{w}, \overline{w}), h)$ and $h \mapsto j^0(x, T\overline{w}(x), h)$ are subadditive, $z \mapsto \inf\ G(\overline{w}, z)$ is convex for all $\overline{w} \in K$, T is additive and $(H_G)(1)$ we get

$$0 \leq \sigma(\mathcal{A}(\overline{w}, \overline{w}), \sum_{i \in \mathcal{I}} \lambda_i w_i - \sum_{i \in \mathcal{I}} \lambda_i \overline{w}) + \sum_{i \in \mathcal{I}} \lambda_i \inf\ G(\overline{w}, w_i - \overline{w}) +$$

$$+\int_\Omega j^0(x, T\overline{w}(x), \sum_{i\in\mathcal{I}} T(\lambda_i w_i)(x) - \sum_{i\in\mathcal{I}} T(\lambda_i \overline{w})(x))dx < 0,$$

which is absurd. Therefore, T_1 is KKM-mapping.

Step 3. We prove that $\bigcap\limits_{w\in K} \overline{T_1(w)} \neq \emptyset$. Here, $\overline{T_1(w)}$ is the closure of $T_1(w)$. Indeed, from Step 1, we have that $T_1(u) \subseteq K_0$. Since K_0 is compact, $\overline{T_1(u_0)}$ is also compact. Using the Step 2 and applying Lemma 2.4., we obtain that $\bigcap\limits_{w\in K} \overline{T_1(w)} \neq \emptyset$.

Step 4. $\bigcap\limits_{w\in K} T_1(w) = \bigcap\limits_{w\in K} T_2(w)$.

(a) Let $u \in \bigcap\limits_{w\in K} T_1(w)$ i.e. $\sigma(\mathcal{A}(u,u), w-u) + \inf G(u, w-u) + \int\limits_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0,\ \forall w \in K$. From (H_{coer}), we have that $u \in K_0$. From the $(H_{\mathcal{A}})(4)$, we can write that

$$\sigma(\mathcal{A}(w,u), w-u) + \inf G(u, w-u) + \int\limits_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0,\ \forall w \in K,$$

i.e. $u \in \bigcap\limits_{w\in K} T_2(w)$.

(b) Let $u \in \bigcap\limits_{w\in K} T_2(w)$, i.e. $\sigma(\mathcal{A}(v,u), v-u) + \inf G(u, v-u) + \int\limits_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall v \in K$. Let $v \in K$ be an arbitrary element. Let $v_t = tv + (1-t)u,\ t \in [0,1]$. Clearly, $v_t \in K$. We have

$$\sigma(\mathcal{A}(v_t,u), v_t-u) + \inf G(u, v_t-u) + \int_\Omega j^0(x, Tu(x), Tv_t(x) - Tu(x))dx \geq 0,\ \forall t \in [0,1].$$

From the linearity of T, we have that

$$\sigma(\mathcal{A}(v_t,u), t(v-u)) + \inf G(u, t(v-u)) +$$

$$+\int_\Omega j^0(x, Tu(x), t(Tv(x) - Tu(x)))dx \geq 0,\ \forall t \in [0,1].$$

From the (H_G) (4) and from the fact that $j^0_y(x, Tu(x), \cdot)$ is positive homogeneous, we obtain

$$\sigma(\mathcal{A}(v_t,u), v-u) + \inf G(u, v-u) + \int_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall t \in (0,1].$$

Using $(H_{\mathcal{A}})$ (3), we have that $\limsup\limits_{t\to 0^+} \sigma(\mathcal{A}(v_t,u), v-u) \leq \sigma(\mathcal{A}(u,u), v-u)$. Therefore,

$$\sigma(\mathcal{A}(u,u), v-u) + \inf G(u, v-u) + \int_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0.$$

Since $v \in K$ was arbitrary, u is a solution for (P). $\square$

Step 5. $\bigcap\limits_{w\in K} \overline{T_1(w)} = \bigcap\limits_{w\in K} T_2(w)$.

Clearly, $\bigcap_{w\in K} T_2(w) \subseteq \bigcap_{w\in K} \overline{T_1(w)}$ from Step 4. Conversely, let $v \in \bigcap_{w\in K} \overline{T_1(w)}$. We prove that $v \in \bigcap_{w\in K} T_2(w)$. Since $\overline{T_1(u_0)} \subset K_0$, we have that $\bigcap_{w\in K} \overline{T_1(w)} \subseteq K_0$. Therefore, $v \in K_0 \cap \overline{T_1(w)},\ \forall w \in K$.

Now, let $u \in K$ be a fixed element. Since $v \in \overline{T_1(u)}$, there exists a sequence $\{v_n\}$ from $T_1(u)$ such that $v_n \to v$. Since $v_n \in T_1(u)$, we have

$$\sigma(\mathcal{A}(v_n, v_n), u - v_n) + \inf\, G(v_n, u - v_n) + \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \geq 0.$$

From $(H_\mathcal{A})(4)$, we have

$$\sigma(\mathcal{A}(u, v_n), u - v_n) + \inf\, G(v_n, u - v_n) + \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \geq 0.$$

From $(H_\mathcal{A})(1)$ and (2), applying Lemma 2.1 we obtain that $v \mapsto \sigma(\mathcal{A}(u, v), u - v)$ is usc, therefore

$$\limsup_{n\to\infty} \sigma(\mathcal{A}(u, v_n), u - v_n) \leq \sigma(\mathcal{A}(u, v), u - v).$$

From Lemma 2.2 (with $F = G,\ Y := K \times X,\ Z := \mathbb{R},\ f((y_1, y_2), z) = -z$, where $z \in G(y_1, y_2)$) and $(H_G)(3)$ we have that $v \mapsto \inf\, G(v, u - v)$ is usc, therefore

$$\limsup_{n\to\infty} \inf\, G(v_n, u - v_n) \leq \inf\, G(v, u - v).$$

Using the Lemma 2.3 we get the following inequality

$$\limsup_{n\to\infty} \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \leq \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx.$$

Summarizing the above relations, we get

$$\sigma(\mathcal{A}(u, v), u - v) + \inf\, G(v, u - v) +$$

$$+ \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx \geq 0,$$

i.e. $v \in T_2(u)$. Since u was arbitrary, we have that $v \in \bigcap_{u\in K} T_2(u)$.

Step 6. From Steps 3, 4 and 5, we have that $\bigcap_{w\in K} T_1(w) \neq \emptyset$, which means that u is a solution for (P).□

Remark 3.1 If K is compact in the above theorem, the hypothesis (H_{coer}) can be omitted.

4. **Applications**

As a first application, we can deduce easily the Schauder fixed point theorem from Theorem 3.1. on Banach spaces. For the completeness, we give the proof.

Corollary 4.1 *Let K be a compact, convex subset of a Banach space X and $f : K \to K$ be a continuous function. Then f has a fixed point.*

Proof. Let $\mathcal{A} \equiv 0$, $j \equiv 0$, $T \equiv 0$ and $G : K \times X \rightsquigarrow \mathbb{R}$ defined by $G(u, v) = [\|u + v - f(u)\| - \|u - f(u)\|, \infty)$.

We verify (H_G). Clearly, $G(u, 0) = [0, \infty) = \mathbb{R}_+$ and $v \rightsquigarrow G(u, v)$ is convex, $\forall\, v \in K$. Since f is continuous, the function $(u, x) \mapsto \|u + x - f(u)\| - \|u - f(u)\|$ is continuous also. Therefore, it's easy to prove that $(u, x) \rightsquigarrow G(u, x)$ is lsc on $K \times X$. The subhomogeneity of $G(u, \cdot)$ for $t = 0$ and $t = 1$ is trivial. Otherwise, this follows from the triangle inequality. Therefore, from Theorem 3.1 it follows that there exists $u_0 \in K$ such that

$$[\|v - f(u_0)\| - \|u_0 - f(u_0)\|, \infty) = G(u_0, v - u_0) \subseteq \mathbb{R}_+,\ \forall\, v \in K.$$

In particular, we have $\|v - f(u_0)\| - \|u_0 - f(u_0)\| \geq 0,\ \forall\, v \in K$. Let $v := f(u_0)$. We have $-\|u_0 - f(u_0)\| \geq 0$, i.e. $u_0 = f(u_0)$. □

Corollary 4.2 (Brouwer fixed point theorem) *Let $f : K \to K$ be a continuous function, K being a compact, convex subset of $\mathbb{R}^n$. Then f has a fixed point.*

Corollary 4.3 [12, Theorem 1.] *Let K be a compact and convex subset of a Banach space X and j and T satisfying (H_j) and (H_T) respectively. If the operator $A : K \to X^*$ is w^*-demicontinuous, then there exists $u \in K$ such that*

$$(PPFR) \qquad \langle Au, v - u\rangle + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall v \in K.$$

Proof. Let $\mathcal{A} : K \times K \rightsquigarrow X^*$ defined by $\mathcal{A}(v, u) = \{A(u)\},\ \forall u, v \in K$ and $G \equiv 0$. Let $v \in K$ be fixed. From Remark 2.1, $\mathcal{A}(v, \cdot)$ is udc on K (with bounded values). Therefore, $(H_{\mathcal{A}})$ holds. Since $\sigma(\mathcal{A}(u, u), v - u) = \langle Au, v - u\rangle$, the assertion follows easily from Theorem 3.1. □

The following result is of Browder's type, see [2].

Corollary 4.4 *Let K be a convex, closed subset of a Banach space, $\mathcal{A} : K \times K \rightsquigarrow X^*$ be an operator satisfying $(H_{\mathcal{A}})$. Suppose that there exists a compact subset $K_0 \subset K$ and $u_0 \in K$ such that $\sigma(\mathcal{A}(u, u), u_0 - u) < 0,\ \forall u \in K \setminus K_0$. Then* *there exists* $u \in K$ *such that*

$$\sigma(\mathcal{A}(u,u), v-u) \geq 0, \ \forall v \in K.$$

Proof. We apply Theorem 3.1 for $G \equiv 0$, $j \equiv 0$ and $T \equiv 0$. $\square$

Remark 4.1 Similar results were obtained by Y-Q. Chen in [3] and by A. M. Croicu and I. Kolumbán in [5].

Finally let X_1 and X_2 two Banach spaces, $K_1 \subseteq X_1$, $K_2 \subseteq X_2$ two nonempty closed, convex sets. Let $F_i : K_1 \times K_2 \to X_i^*$, $i = 1,2$ two operators. Our aim is to give existence result for the following problem:

Find $(u_1, u_2) \in K_1 \times K_2$ such that

$$(NP) \qquad \langle F_1(u_1,u_2), x-u_1\rangle \geq 0, \ \forall x \in K_1$$

$$\langle F_2(u_1,u_2), y-u_2\rangle \geq 0, \ \forall y \in K_2.$$

The above problem is originated from the Nash equilibrium points, see [10] and [9].

Theorem 4.1 *Suppose that*

(i) *for every* $x_i \in K_i$, $i = 1,2$ *the mappings* $F_1(\cdot, x_2) : K_1 \to X_1^*$ *and* $F_2(x_1, \cdot) : K_2 \to X_2^*$ *are monotones and udc on the line segments in* K_1 *respective* K_2 (*in particular hemicontinuous*);

(ii) *for every* $x_i \in K_i$, $i = 1,2$ *the mappings* $F_1(\cdot, x_2) : K_1 \to X_1^*$ *and* $F_2(x_1, \cdot) : K_2 \to X_2^*$ *are* w^**-demicontinuous*;

(iii) *there exist* $K_i^0 \subseteq K_i$, $i = 1,2$ *compact sets and* $x_i^0 \in K_i^0$ *such that for every* $(x_1,x_2) \in (K_1 \times K_2) \setminus (K_1^0 \times K_2^0)$

$$\langle F_1(x_1,x_2), x_1^0 - x_1\rangle + \langle F_2(x_1,x_2), x_2^0 - x_2\rangle < 0.$$

Then (NP) *has at least a solution.*

Proof. First let $j \equiv 0$, $G \equiv 0$ and $T \equiv 0$ in Theorem 3.1. Moreover, let $X := X_1 \times X_2$, $K := K_1 \times K_2$ and $\mathcal{A} : K \times K \rightsquigarrow X^*$ be a single-valued map, defined by

$$\mathcal{A}((x,y),(z,t)) = (F_1(x,t), F_2(z,y)), \ \forall (x,y), \ (z,t) \in K.$$

Clearly, $\mathcal{A}$ satisfies $(H_{\mathcal{A}})$. Let $K_0 := K_1^0 \times K_2^0$ and $u_0 := (x_1^0, x_2^0) \in K_0$. The K_0 and u_0 satisfy the (H_{coer}) condition from Theorem 3.1. Therefore, there exists $u = (u_1, u_2) \in K$ such that $\langle \mathcal{A}(u, u), w - u \rangle \geq 0, \ \forall w \in K$. This is equivalent with

$$\langle F_1(u_1, u_2), w_1 - u_1 \rangle + \langle F_2(u_1, u_2), w_2 - u_2 \rangle \geq 0, \ \forall w_i \in K_i, \ i = \overline{1, 2}.$$

Substituting $w_2 := u_2$ and $w_1 := u_1$ respectively, we obtain that $u = (u_1, u_2) \in K$ is a solution for (NP). $\square$

Remark 4.2 If K_1 and K_2 are compact sets, the hypothesis (iii) from the above theorem can be omitted.

Remark 4.3 From the above theorem we obtain also the Brouwer fixed point theorem (see Corollary 4.2) choosing $K = K_1 = K_2$, $F_1(u_1, u_2) = -f(u_1) + u_2$ and $F_2(u_1, u_2) = u_2 - u_1$.

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