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MAXIMAL SETS ON A HYPERSPHERE

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Abstract. It is studied the problem of the maximum number of points situated on a hypersphere of radius 1 with the property that the distances between any two points is at least r. It is solved the case $r = \sqrt{2}$.

1. **Introduction**

The goal of this paper is to find the maximum number of points of hypersphere, such that the distances between every two points is great that a given number. The solution of the problem in the general case is very difficult. we solved the problem in a remarcable particular case.

Let $S_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n |\ x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in $\mathbb{R}^n$. For every real number $r \in [0, 2]$ we define the natural numbers $N(n, r)$ and $\overline{N}(n, r)$ by: $N(n, r)$ is the maximum number of elements of a set $M \subseteq S_{n-1}$ with the property that the distance $d(A, B)$ between every two points $A, B \in M$ satisfies the relation $d(A, B) > r$.

$\overline{N}(n, r)$ is the maximum numbers of elements of a set $M \subseteq S_{n-1}$ with the property $d(A, B) \geq r$ for every $A, B \in M$.

We think that the determination of a general expression for the functions $N, \overline{N} : \mathbb{N}^* \times [0, 2] \to \mathbb{N}$ is not possible. We solve the problem for $r = \sqrt{2}$. The following properties of N and $\overline{N}$ are easy to verify.

1. $N(n, r) \leq \overline{N}(n, r)$;
2. $N(n, r) \leq N(n + 1, r)$;
3. $\overline{N}(n, r) \leq \overline{N}(n + 1, r)$;
4. $N(n, r_1) \leq N(n, r_2)$ for $n_1 > n_2$;
5. $\overline{N}(n, r_1) \leq \overline{N}(n, r_2)$ for $r_1 > r_2$;
6. $N(1, r) = 2$ for $r \in [0, 2)$.

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7. $N(1,2) = 0$;

8. $\overline{N}(1,r) = 2$;

9. $\overline{N}(2,r) = \left[\dfrac{\pi}{\arcsin\dfrac{r}{2}}\right]$;

10. $N(2,r) = \overline{N}(2,r)$, if $\dfrac{\pi}{\arcsin\dfrac{r}{2}} \notin \mathbb{N}$;

11. $N(2,r) = \overline{N}(2,r) - 1$, if $\dfrac{\pi}{\arcsin\dfrac{r}{2}} \in \mathbb{N}$;

12. $N(n,2) = 0$, $\overline{N}(n,2) = 2$.

Theorem 1. *For every natural number $n \geq 1$ we have*

$$N(n,\sqrt{2}) = n+1 \text{ and } \overline{N}(n,\sqrt{2}) = 2n.$$

Proof. The distance between the points $X = (x_1,\ldots,x_n)$ and $Y = (y_1,\ldots,y_n)$ is:

$$d^2(X,Y) = \sum_{k=1}^{n}(x_k - y_k)^2.$$

We have

$$d(X,Y) > \sqrt{2} \Leftrightarrow d^2(X,Y) > 2$$

$$\Leftrightarrow \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 + 2\sum_{k=1}^{n} x_k y_k > 2$$

$$\Leftrightarrow \sum_{k=1}^{n} x_k y_k < 0 \tag{1}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1,0,\ldots,0).$$

For $X = A_1$, condition (1) for implies $y_1 > 0,\ \forall\, Y \in M_n$.

Let $X = (x_1,\overline{X})$, $Y = (y_1,\overline{Y}) \in M_n \setminus \{A_1\}$, $\overline{X},\overline{Y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n} x_k y_k < 0 \Rightarrow x_1 y_1 + \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0 \Leftrightarrow \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\overline{x}_k}{\sqrt{\sum \overline{x}_k^2}}, \quad y'_k = \frac{\overline{y}_k}{\sqrt{\sum \overline{y}_k^2}},$$

therefore

$$(x'_1, \ldots, x'_{n-1}), (y'_1, \ldots, y'_{n-1}) \in S_{n-2}$$

and verify condition (1).

If a_n is the search number of points in $\mathbb{R}^n$, we obtain $a_n \leq 1 + a_{n-1}$ and $a_1 = 2$ implies that $a_n \leq n + 1$.

We show that $a_n = n+1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$A_1 = (-1, 0, 0, 0, \ldots, 0, 0)$$
$$A_2 = \left(\frac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right)$$
$$A_3 = \left(\frac{1}{n}, \frac{1}{n-1}c_1, -c_2, 0, \ldots, 0, 0\right)$$
$$A_4 = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-1}c_2, -c_3, \ldots, 0, 0\right)$$
$$A_{n-1} = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, -c_{n-2}, 0\right)$$
$$A_n = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, -c_{n-1}\right)$$
$$A_{n+1} = \left(\frac{1}{n}, \frac{1}{n-1}c_1, \frac{1}{n-2}c_2, \frac{1}{n-3}c_3, \ldots, \frac{1}{2}c_{n-2}, c_{n-1}\right)$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}, \quad k = \overline{1, n-1}.$$

We have

$$\sum_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0 \text{ and } \sum_{k=1}^{n} x_k^2 = 1, \ \forall\ X, Y \in \{A_1, \ldots, A_{n+1}\}.$$

This points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points are equal to

$$d = \sqrt{2}\sqrt{1 + \frac{1}{n}} > \sqrt{2}.$$

Remark. For $n = 2$ the points form an equilateral triangle in the unit circle; for $n = 3$ the four points from a regular tetrahedron and in $\mathbb{R}^n$ the points from an n dimensional regular simplex.

For the function $\overline{N}$ we have $\overline{N}(1, \sqrt{2}) = 2$.

$$(M = \{-1, 1\}, \quad \overline{N} = (2, \sqrt{2}) = 4), \quad (M = \{(-1, 0), (1, 0), (0, -1), (0, 1)\})$$

By induction, intersecting the hypersphere S_n from $\mathbb{R}^{n+1}$ with the hyperplane $x_{n+1} = 0$ we obtain the hypersphere S_n, which contains a maximal set with $2n$ points and considering the points $(0, \dots, 0, -1)$ and $(0, \dots, 0, 1)$ on S_n we obtain a maximal set with $2(n+1)$ points, hence $\overline{N}(n+1, \sqrt{2}) = 2(n+1)$.

We remark that a maximal set for $\overline{N}(n, \sqrt{2})$ is

$$M = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1), (-1, 0, \dots, 0),$$

$$(0, -1, 0, \dots, 0), \dots, (0, \dots, 0, -1)\}$$

with n distances equal with 2 and the rest of $C_{2n}^2 - n = 2n(n-1)$ distances equal with $\sqrt{2}$.

It is known that every real euclidean n-dimensional space is isomorphic with $\mathbb{R}^n$ and we can transpose the results by the isomorphism.

If $(V, \langle \cdot, \cdot \rangle)$ is an real euclidean space, by the theorem 1, we have the following consequences.

Proposition 1. If the dimension of V is n, then for any $(n+2)$ vectors with norm 1, there exists two with the distances is at most $\sqrt{2}$.

Proposition 2. If the dimension of V is n, then for any $(n+2)$ nonzero vectors there exists two vectors with an acute angle

$$\left(d(X, Y) \leq \sqrt{2}, \quad \|X\| = \|Y\| = 1 \Leftrightarrow \langle X, Y \rangle \geq 0 \Leftrightarrow \widehat{X, Y} \leq \frac{\pi}{2}\right)$$

Proposition 3. If in euclidean space V there exists a set of $(n+1)$ vectors $\{X_1, \dots, X_n, X_{n+1}\}$ with the property $\langle X_i, X_j \rangle < 0$, for any $i \neq j$, $i, j = \overline{1, n}$, then the dimension of V is $\dim V \geq n$.

2. **Applications**

Problem 1. Let $n \in \mathbb{N}^*$ be a natural number. Find all $m \in \mathbb{N}^*$ so that there exists a matrix $A \in \mathcal{M}(m, n)(\mathbb{R})$ with the property that all elements of the matrix $A \cdot A^t$, outside the main diagonal to be negative numbers.

Solution. Denote by $L_1, \dots, L_m \in \mathbb{R}^n$ the lines of matrix A. The element b_{ij} of the matrix $B = A \cdot A^t$ is the inner product $\langle L_i, L_j \rangle$, so the condition is that for $i \neq j$ to have $\langle L_i, L_j \rangle < 0$. From proposition 3 we obtain $m \leq n + 1$.

Problem 2. If $\{P_1, \dots, P_n, P_{n+1}\}$ is a set of polynomials with real coefficients so that:

$$\int_0^1 P_i(x)P_j(x)dx < 0, \text{ for any } i \neq j,$$

then at least one polynomial has the degree at least $(n-1)$.

Solution. Denote by V the real vector space generated by the polynomials $P_1, \dots, P_n, P_{n+1}$, and on V define the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Using the proposition 3, it result that $\dim V \geq n$. If $degP_k < n-1$ for all $k = \overline{1, n+1}$, then V is a subspace in the space of polynomials with degree $\leq n-2$, with the dimension $(n-1)$. We obtain the contradiction $n \leq n-1$.

Problem 3. Show that the inequalities

$$a_1a_2 + b_1b_2 < 0$$

$$a_1a_3 + b_1b_3 < 0$$

$$a_1a_4 + b_1b_4 < 0$$

$$a_2a_3 + b_2b_3 < 0$$

$$a_2a_4 + b_2b_4 < 0$$

$$a_3a_4 + b_3b_4 < 0$$

does not hold simultaneously.

Solution. In euclidean plane $\mathbb{R}^2$ consider the points $A_i(a_i, b_i)$, $i = \overline{1, 4}$. The condition $a_ia_j + b_ib_j < 0$ is equivalent with the angle $\widehat{A_iOA_j} > \frac{\pi}{2}$, which is impossible for every $i \neq j$.

Remark. Another remarkable value for r is $r = 1$. We have not succeed to find $\overline{N}(n, 1)$ but we suppose that $\overline{N}(n, 1) = n(n+1)$.

References

[1] I. Tomescu, *Introduction a la combinatorique*, Ed. Tehnică, Bucureşti, 1972.
[2] I.M. Glaman, Iu.I. Liubici, *Analiză liniară pe spaţii finit dimensionale*, Ed. Şt. şi Enciclopedică, Bucureşti, 1980.

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