STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 3, September 2002

QUANTITATIVE ESTIMATES FOR SOME LINEAR AND POSITIVE OPERATORS

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Abstract. The purpose of this paper is to establish quantitative estimates for the rate of convergence of some linear and positive operators. The most of them are generated by special functions.

1. **Introduction**

For the Bernstein operator

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0,1], \quad \varphi(x) = \sqrt{x(1-x)}$$

it is well - known that there exists an absolute constant $C > 0$ such that

$$|B_n(f,x) - f(x)| \ \leq \ C\ \omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right), \ x \in [0,1] \tag{1}$$

and

$$\|B_n(f) - f\| \ \leq \ C\ \omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right). \tag{2}$$

(see [2, p. 308, Theorem 3.2] and [3, p. 117, Theorem 9.3.2], respectively). Here

$$\omega_2(f,\delta) = \sup_{0<h\leq\delta} \ \sup_{x,x\pm h\in[0,1]} |f(x+h) - 2f(x) + f(x-h)|$$

is the usual second moduli of smoothness and

$$\omega_2^{\varphi}(f,\delta) = \sup_{0<h\leq\delta} \ \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

Received by the editors: 04.06.2002.

2000 *Mathematics Subject Classification.* 41A10, 41A36.

Key words and phrases. the first and the second moduli of smoothness, the first and the second modulus of smoothness of Ditzian - Totik.

$\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ is the second modulus of smoothness of Ditzian - Totik. Furthermore, we shall use the first and second moduli of smoothness of a function $g : I \to \Re$ as defined by

$$\omega_1(g,\delta) = \sup_{0<h\le\delta} \sup_{x,x+h\in I} |g(x+h) - g(x)|,$$

$$\omega_2(g,\delta) = \sup_{0<h\le\delta} \sup_{x,x\pm h\in I} |g(x+h) - 2g(x) + g(x-h)|,$$

and the following Ditzian - Totik type moduluses of smoothness:

$$\omega_1^{\varphi}(g,\delta) = \sup_{0<h\le\delta} \sup_{x\pm h\varphi(x)\in[0,1]} \left| g\left(x + \frac{h}{2}\varphi(x)\right) - g\left(x - \frac{h}{2}\varphi(x)\right) \right|,$$
$$g \in C[0,1], \varphi(x) = \sqrt{x(1-x)},$$

$$\omega_2^{\varphi}(g,\delta)_{\infty} = \sup_{0<h\le\delta} \sup_{x\pm h\varphi(x)\in[0,\infty)} |g(x+h\varphi(x)) - 2g(x) + g(x-h\varphi(x))|,$$
$$g \in C_B[0,\infty), \varphi(x) = \sqrt{x},$$

where $C_B[0,\infty)$ denotes the set of all bounded and continuous functions on $[0,\infty)$.

The aim of this paper is to establish pointwise and global uniform quantitative estimates for some linear and positive operators using the above mentioned moduluses of smoothness, obtaining estimates similar to (1) and (2). These operators are the following:

1. *Stancu's operator* [9]:

$$S_n^{\alpha}(f,x) = \sum_{k=0}^{n} w_{n,k}(x,\alpha) f\left(\frac{k}{n}\right), \; f \in C[0,1], \; \alpha \ge 0$$

and

$$w_{n,k}(x,\alpha) = \begin{pmatrix} n \\ k \end{pmatrix} \frac{\prod_{i=o}^{k-1} (x + i\alpha) \prod_{j=0}^{n-k-1} (1 - x + j\alpha)}{\prod_{r=0}^{n-1} (1 + r\alpha)};$$

2. *Lupaş' operator* [5]:

$$\bar{B}_n(f,x) = \frac{1}{B(nx, n-nx)} \cdot \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\,dt, \qquad x \in (0,1)$$

and $\bar{B}_n(f,0) = f(0), \bar{B}_n(f,1) = f(1)$;

3. *Miheşan's operators* [7]:

a) if ${}_2F_1(a,b,c,z)$ is the hypergeometric function and in the integral form

$$ {}_2F_1(a,b,c,z) = \frac{1}{B(a,c-a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\, dt, $$

$a,b,c,z \in \Re$, $|z|<1$, $c \neq 0,-1,-2,\ldots$ and $c>a>0$ then

$$ F_n^*(f,x) = \sum_{k=0}^{n} \frac{{}_2F_1\left(\frac{x}{\alpha}+k, b, \frac{1}{\alpha}+n, z\right)}{{}_2F_1\left(\frac{x}{\alpha}, b, \frac{1}{\alpha}, z\right)} \cdot w_{n,k}(x,\alpha) \cdot f\left(\frac{k}{n}\right), $$

$f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $b \geq 0$, $0 \leq z < 1$;

b) if ${}_1F_1(a,c,z)$ is the confluent hypergeometric function of the first kind and in the integral form

$$ {}_1F_1(a,c,z) = \frac{1}{B(a,c-a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\, dt, $$

$a,c,z \in \Re$, $c \neq 0,-1,-2,\ldots$ and $c>a>0$ then

$$ \mathcal{F}_n^*(f,x) = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{\int_0^1 t^{\frac{x}{\alpha}+k-1}(1-t)^{\frac{1-x}{\alpha}+n-k-1}e^{zt}\, f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}e^{zt}\, dt}, $$

$f \in C[0,1]$, $x \in [0,1]$, $\alpha > 0$, $z \geq 0$;

c)

$$ \begin{aligned} L_n^*(f,x) &= e^{-na} \sum_{k=0}^{\infty} \frac{(na)^k}{k!} \cdot \frac{nx(nx+1)\ldots(nx+k-1)}{na(na+1)\ldots(na+k-1)} \cdot \\ &\quad \cdot {}_1F_1(na-nx, na+k, na) \cdot f\left(\frac{k}{n}\right), \end{aligned} $$

$f \in C[0,\infty)$, $x \in [0,a]$;

d)

$$ \begin{aligned} \tilde{L}_n(f,x) &= \left(\frac{b+c}{c}\right)^{-nx} \sum_{k=0}^{\infty} \frac{b(b+1)\ldots(b+k-1)}{c(c+1)\ldots(c+k-1)} \cdot \\ &\quad \cdot \frac{nx(nx+1)\ldots(nx+k-1)}{k!} \cdot \left(\frac{b}{b+c}\right)^k \cdot \\ &\quad \cdot {}_2F_1\left(nx+k, c-b, c+k, \frac{b}{b+c}\right) \cdot f\left(\frac{k}{n}\right), \end{aligned} $$

$f \in C[0,\infty)$, $x \in [0,\infty)$ $0<b<c$.

4. Furthermore, we define a *generalization of Goodmann and Sharma's operator* as follows:

$$ U_n^{\alpha}(f,x) = f(0)w_{n,0}(x,\alpha) + f(1)w_{n,n}(x,\alpha) + $$

$$+\sum_{k=1}^{n-1} w_{n,k}(x,\alpha) \int_0^1 (n-1)\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} t^{k-1}(1-t)^{n-1-k} f(t)\, dt,$$

$f \in C[0,1]$, $\alpha \geq 0$.

Remark 1. *a*) For $b = c$ we obtain

$$\tilde{L}_n(f,x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{nx(nx+1)\dots(nx+k-1)}{2^k\ k!}\ f\left(\frac{k}{n}\right).$$

This operator was introduced by Lupaş in [6].

b) Here we mention that throughout this paper C denotes absolute constant and not necessarily the same at each occurrence.

2. **Theorems**

Before we state our results let us observe that the operators introduced in 1), 2), 3*a*), 3*b*) and 4) are generated by special functions. Indeed, if $\mathcal{B}_\alpha : C[0,1] \to C[0,1]$,

$$\mathcal{B}_\alpha(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt}\ ;$$

$F_{b,z}^{\alpha} : C[0,1] \to C[0,1]$,

$$F_{b,z}^{\alpha}(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-b} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-b}\, dt}$$

and $\mathcal{F}_z^{\alpha} : C[0,1] \to C[0,1]$,

$$\mathcal{F}_z^{\alpha}(f,x) = \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} e^{zt} f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} e^{zt}\, dt}$$

then, in view of [9, Theorem 3.1] and [7, Propoziţia 2.18 and Propoziţia 2.19] we have

$$S_n^{\alpha}(f,x) = \mathcal{B}_\alpha(B_n(f),x); \tag{3}$$

$$U_n^{\alpha}(f,x) = \mathcal{B}_\alpha(U_n(f),x), \tag{4}$$

where

$$U_n(f,x) = f(0)(1-x)^n + f(1)x^n +$$

$$+\sum_{k=1}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k} \cdot \int_0^1 (n-1)\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} t^{k-1}(1-t)^{n-1-k}\ f(t)\, dt,$$

$f \in C[0,1]$, is the *Goodman - Sharma's operator* [8];

$$\bar{B}_n(f,x) = \mathcal{B}_{\frac{1}{n}}(f,x); \tag{5}$$

$$F_n^*(f,x) = F_{b,z}^{\alpha}(B_n(f),x) \tag{6}$$

and

$$\mathcal{F}_n^*(f,x) = \mathcal{F}_z^{\alpha}(B_n(f),x). \tag{7}$$

Furthermore, let us consider the following notations

$$\begin{aligned}\beta(n,x,\alpha,b,z) \quad &= \quad \frac{1}{n}\ (1-z)^{-(b+1)} \cdot \frac{x(1-x)}{1+\alpha} + (1-z)^{-(b+1)} \cdot \frac{\alpha x(1-x)}{1+\alpha} + \\ &\quad + 2\ (1-z)^{-(b+1)} \cdot \left(1-(1-z)^{2(b+1)}\right) x^2,\end{aligned}$$

$x \in [0,1]$, $\alpha > 0$, $b \geq 0$, $0 \leq z < 1$;

$$\gamma(n,x,\alpha,z) = \frac{1}{n}\ e^z \cdot \frac{x(1-x)}{1+\alpha} + e^z \cdot \frac{\alpha x(1-x)}{1+\alpha} + 2\ e^z\ (1-e^{-2z})\ x^2,$$

$x \in [0,1]$, $\alpha > 0$, $z \geq 0$;

$$\begin{aligned}\beta'(n,\alpha,b,z) \quad &= \quad \frac{1}{4n}\ (1-z)^{-(b+1)} \cdot \frac{1}{1+\alpha} + \frac{1}{4}\ (1-z)^{-(b+1)} \cdot \frac{\alpha}{1+\alpha} + \\ &\quad + 2\ (1-z)^{-(b+1)} \left(1-(1-z)^{2(b+1)},\right)\end{aligned}$$

$\alpha > 0$, $b \geq 0$, $0 \leq z < 1$ and

$$\gamma'(n,\alpha,z) = \frac{1}{4n}\ e^z \cdot \frac{1}{1+\alpha} + \frac{1}{4}\ e^z \cdot \frac{\alpha}{1+\alpha} + 2\ e^z(1-e^{-2z}),$$

$\alpha > 0$, $z \geq 0$, respectively.

The next theorem contains the local approximation results for the above mentioned operators:

Theorem 1. *For all $f \in C[0,1]$ we have*

a) $|S_n^{\alpha}(f,x) - f(x)| \leq C\ \omega_2\left(f, \sqrt{\frac{1+n\alpha}{n(1+\alpha)} \cdot x(1-x)}\ \right), \quad x \in [0,1];$

b) $|U_n^{\alpha}(f,x) - f(x)| \leq C\ \omega_2\left(f, \sqrt{\left(\frac{2}{n+1} + \alpha\right) \cdot \frac{x(1-x)}{1+\alpha}}\ \right), \quad x \in [0,1];$

c) $|\bar{B}_n(f,x) - f(x)| \leq C\ \omega_2\left(f, \sqrt{\frac{x(1-x)}{n+1}}\ \right), \quad x \in [0,1];$

d) $|F_n^*(f,x) - f(x)| \leq C\ \omega_1\left(f, \sqrt{\beta(n,x,\alpha,b,z)}\ \right), \quad x \in [0,1];$

e) $|\mathcal{F}_n^*(f,x) - f(x)| \leq C\ \omega_1\left(f, \sqrt{\gamma(n,x,\alpha,z)}\ \right), \quad x \in [0,1].$

For all $f \in C[0,\infty)$ we have

$$f)\ |L_n^*(f,x)-f(x)| \ \le\ C\ \omega_2\left(f, \sqrt{\tfrac{x}{n}+\tfrac{x(a-x)}{na+1}}\ \right), \quad x\in[0,a];$$
$$g)\ |\tilde{L}_n(f,x)-f(x)| \ \le\ C\ \omega_2\left(f, \sqrt{\tfrac{x}{n}+\tfrac{nx^2(c-b)+c(b+1)x}{nb(c+1)}}\ \right), \quad x\in[0,\infty).$$

With the notations $\|f\| = \sup\ \{|f(x)| : x \in [0,1]\}$ for $f \in C[0,1]$ and $\|f\|_\infty = \sup\ \{|f(x)| : x \ge 0\}$ for $f \in C_B[0,\infty)$, the global approximation results can be included in the following theorem:

Theorem 2. *For all $f \in C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$ we have*

$$a)\ \|S_n^\alpha(f)-f\| \ \le\ C\ \omega_2^\varphi\left(f, \sqrt{\tfrac{1+n\alpha}{n(1+\alpha)}}\ \right);$$
$$b)\ \|U_n^\alpha(f)-f\| \ \le\ C\ \omega_2^\varphi\left(f, \sqrt{\tfrac{1}{1+\alpha}\left(\tfrac{2}{n+1}+\alpha\right)}\ \right);$$
$$c)\ \|\bar{B}_n(f)-f\| \ \le\ C\ \left\{\omega_2^\varphi\left(f, \sqrt{\tfrac{1}{n}}\ \right)+\omega_2^\varphi\left(f, \sqrt{\tfrac{2}{n+1}}\ \right)\right\};$$
$$d)\ \|F_n^*(f)-f\| \ \le\ C\ \omega_1^\varphi\left(f, \sqrt[4]{\beta'(n,\alpha,b,z)}\ \right),$$
$$e)\ \|\mathcal{F}_n^*(f)-f\| \ \le\ C\ \omega_1^\varphi\left(f, \sqrt[4]{\gamma'(n,\alpha,z)}\ \right).$$

For all $f \in C_B[0,\infty)$ and $\varphi(x) = \sqrt{x}$ we have

$$f)\ \|\tilde{L}_n(f)-f\|_\infty \ \le\ C\ \omega_2^\varphi\left(f, \sqrt{\tfrac{1}{n}}\ \right)_\infty, \quad \text{when } b=c.$$

3. **Proofs**

Proof of Theorem 1. The statements *a*), *b*), *c*) can be proved with the same method, therefore we shall give the proof for *b*). In fact *a*) was proved in [4, Lemma 4], when $0 < \alpha(n)\cdot n \le 1$ $(n = 1,2,\ldots)$, obtaining the estimate (1) for S_n^α.

At first, let us observe that U_n^α preserves the linear functions. Indeed, by (4), [8, (2.2)] and definition of $\mathcal{B}_\alpha$ we get

$$\begin{aligned} U_n^\alpha(u-x,x) &= \mathcal{B}_\alpha\left(U_n(u-x,t),x\right) \\ &= \mathcal{B}_\alpha\left(U_n((u-t)+(t-x),t),x\right) \\ &= \mathcal{B}_\alpha\left(t-x,x\right) = \frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)} - x = 0. \qquad (8)\end{aligned}$$

Moreover, by (4) and [8, (2.2) - (2.3)] we obtain

$$\begin{aligned} U_n^\alpha\left((u-x)^2,x\right) &= \mathcal{B}_\alpha\left(U_n((u-x)^2,t),x\right) \\ &= \mathcal{B}_\alpha\left(U_n((u-t)^2+2(u-t)(t-x)+(t-x)^2,t),x\right) \\ &= \mathcal{B}_\alpha\left(U_n((u-t)^2,t)+(t-x)^2,x\right) \\ &= \mathcal{B}_\alpha\left(\frac{2t(1-t)}{n+1}+t^2-2xt+x^2,x\right) \\ &= \frac{2}{n+1}\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \end{aligned}$$

$$-\,2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}=\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}. \tag{9}$$

Finally, by (4) and [8, (2.4)] we get

$$\begin{aligned} |U_n^\alpha(f,x)| &\le \mathcal{B}_\alpha\left(|U_n(f,t)|,x\right) \\ &\le \|U_n(f)\|\cdot\mathcal{B}_\alpha(1,x)=\|U_n(f)\|\le\|f\|. \end{aligned}$$

Thus

$$\|U_n^\alpha(f)\| \;\le\; \|f\|. \tag{10}$$

Now, let $g\in C^2[0,1]$. By Taylor's formula we have

$$g(u)=g(x)+(u-x)g'(x)+\int_x^u (u-v)g''(v)\,dv. \tag{11}$$

Hence, by (8) we have

$$U_n^\alpha(g,x)-g(x)=U_n^\alpha\left(\int_x^u (u-v)g''(v)\,dv\;,\;x\right).$$

Then, by (9)

$$\begin{aligned} |U_n^\alpha(g,x)-g(x)| &\le U_n^\alpha\left(\left|\int_x^u |u-v|\cdot|g''(v)|\,dv\right|,x\right) \\ &\le U_n^\alpha\left((u-x)^2,x\right)\cdot\|g''\|=\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\cdot\|g''\|. \end{aligned}$$

Hence, by (10)

$$\begin{aligned} |U_n^\alpha(f,x)-f(x)| &\le |U_n^\alpha(f-g,x)-(f-g)(x)|+|U_n^\alpha(g,x)-g(x)| \\ &\le 2\,\|f-g\|+\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\cdot\|g''\|. \end{aligned}$$

Thus

$$\begin{aligned} |U_n^\alpha(f,x)-f(x)| &\le 2\inf_g\left\{\|f-g\|+\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\cdot\|g''\|\right\} \\ &= 2\,K_2\left(f,\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}\right). \end{aligned}$$

Using the equivalence between $K_2(f,\delta)$ and $\omega_2(f,\sqrt{\delta}\,)$ (see [2, p. 177, Theorem 2.4]) we obtain that

$$|U_n^\alpha(f,x)-f(x)| \le C\,\omega_2\left(f,\sqrt{\left(\frac{2}{n+1}+\alpha\right)\cdot\frac{x(1-x)}{1+\alpha}}\right).$$

In view of [7, Lemma 2.22] and [7, (2.50)] we have that L_n^* and $\tilde{L}_n$ preserve the linear functions and

$$L_n^*((u-x)^2,x)=\frac{x}{n}+\frac{x(a-x)}{na+1}$$

and

$$\tilde{L}_n((u-x)^2,x)=\frac{x}{n}+\frac{nx^2(c-b)+c(b+1)x}{nb(c+1)},$$

respectively. Using the same idea as above, we get

$$|L_n^*(f,x)-f(x)| \le C\,\omega_2\left(f,\sqrt{\frac{x}{n}+\frac{x(a-x)}{na+1}}\right)$$

and

$$|\tilde{L}_n(f,x)-f(x)| \le C\,\omega_2\left(f,\sqrt{\frac{x}{n}+\frac{nx^2(c-b)+c(b+1)x}{nb(c+1)}}\right).$$

Thus we have proved the f) and g) statements.

For d) and e) we use the standard method:

$$|f(u)-f(x)|\le\omega_1(f,|u-x|)\le(1+\delta^{-2}(u-x)^2)\,\omega_1(f,\delta),$$

where $u,x\in[0,1]$ and $\delta>0$. Hence

$$|F_n^*(f,x)-f(x)| \le \left[1+\delta^{-2}\cdot F_n^*((u-x)^2,x)\right]\cdot\omega_1(f,\delta) \tag{12}$$

and

$$|\mathcal{F}_n^*(f,x)-f(x)| \le \left[1+\delta^{-2}\cdot\mathcal{F}_n^*((u-x)^2,x)\right]\cdot\omega_1(f,\delta), \tag{13}$$

respectively. Therefore we have to estimate $F_n^*((u-x)^2,x)$ and $\mathcal{F}_n^*((u-x)^2,x)$. These estimates can be found by (6) and (7), if we determine an upper and lower bound for $F_{b,z}^\alpha(f,x)$ and $\mathcal{F}_z^\alpha(f,x)$, respectively.

Let $b > 0$ and $f \geq 0$ on $[0,1]$ (for $b = 0$ we receive back the Stancu's operator using the definition of F_n^*). Then there exists a natural number m such that $m \leq b < m+1$. From $0 < 1-z \leq 1-zt \leq 1$ ($0 \leq z < 1$, $0 \leq t \leq 1$) we obtain $(1-zt)^{m+1} < (1-zt)^b \leq (1-zt)^m$. Hence

$$\begin{aligned} F_{b,z}^{\alpha}(f,x) &\leq \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m-1}\, f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m}\, dt} \\ &= \frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot(1-zt)^{-2}\cdot f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot(1-zt)^{-1}\, dt} \\ &\leq (1-z)^{-2}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\cdot f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(1-zt)^{-m+1}\, dt}. \end{aligned}$$

Using $(m-1)-$ times the last inequality, we obtain

$$\begin{aligned} F_{b,z}^{\alpha}(f,x) &\leq (1-z)^{-m-1}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \\ &\leq (1-z)^{-(b+1)}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \end{aligned} \tag{14}$$

In similar way

$$F_{b,z}^{\alpha}(f,x) \geq (1-z)^{b+1}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt}. \tag{15}$$

Analogously, from $1 \leq e^{zt} \leq e^z$ ($z \geq 0$, $0 \leq t \leq 1$) and $f \geq 0$ on $[0,1]$, we get

$$\mathcal{F}_z^{\alpha}(f,x) \leq e^z\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt} \tag{16}$$

and

$$\mathcal{F}_z^{\alpha}(f,x) \geq e^{-z}\cdot\frac{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}f(t)\, dt}{\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\, dt}, \tag{17}$$

respectively. Now, by (6), (14) and (15) we have

$$
\begin{aligned}
F_n^*((u-x)^2,x) &= \\
&= F_{b,z}^{\alpha}\left(B_n((u-x)^2,t),x\right) \\
&= F_{b,z}^{\alpha}\left(B_n((u-t)^2+2(u-t)(t-x)+(t-x)^2,t),x\right) \\
&= F_{b,z}^{\alpha}\left(B_n((u-t)^2,t)+(t-x)^2,x\right) \\
&= F_{b,z}^{\alpha}\left(\frac{t(1-t)}{n}+t^2-2xt+x^2,x\right) \\
&\le \frac{1}{n}\cdot(1-z)^{-(b+1)}\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+(1-z)^{-(b+1)}\cdot\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \\
&\quad -(1-z)^{b+1}\cdot 2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+(1-z)^{-(b+1)}\cdot x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)} \\
&= \frac{1}{n}\;(1-z)^{-(b+1)}\cdot\frac{x(1-x)}{1+\alpha}+(1-z)^{-(b+1)}\cdot\frac{\alpha x(1-x)}{1+\alpha}+ \\
&\quad +2\;(1-z)^{-(b+1)}\cdot\left(1-(1-z)^{2(b+1)}\right)\;x^2 \\
&= \beta(n,x,\alpha,b,z). \qquad (18)
\end{aligned}
$$

Hence, by (12) and choosing $\delta^2=\beta(n,x,\alpha,b,z)$ we get for $C=2$

$$|F_n^*(f,x)-f(x)|\;\le\;C\;\omega_1\left(f,\sqrt{\beta(n,x,\alpha,b,z)}\;\right).$$

Analogously, by (7), (16) and (17) we have

$$
\begin{aligned}
\mathcal{F}_n^*\left((u-x)^2,x\right) &= \mathcal{F}_z^{\alpha}\left(B_n((u-x)^2,t),x\right) \\
&= \mathcal{F}_z^{\alpha}\left(B_n((u-t)^2,t)+(t-x)^2,x\right) \\
&= \mathcal{F}_z^{\alpha}\left(\frac{t(1-t)}{n}+t^2-2xt+x^2,x\right) \\
&\le \frac{1}{n}\cdot e^z\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}+1\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+e^z\cdot\frac{B\left(\frac{x}{\alpha}+2,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}- \\
&\quad -e^{-z}\cdot 2x\cdot\frac{B\left(\frac{x}{\alpha}+1,\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}+e^z\cdot x^2\cdot\frac{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)} \\
&= \frac{1}{n}\cdot e^z\cdot\frac{x(1-x)}{1+\alpha}+e^z\cdot\frac{\alpha x(1-x)}{1+\alpha}+2e^z(1-e^{-2z})x^2 \\
&= \gamma(n,x,\alpha,z). \qquad (19)
\end{aligned}
$$

Hence, by (13) and choosing $\delta^2=\gamma(n,x,\alpha,z)$ we get for $C=2$

$$|\mathcal{F}_n^*(f,x)-f(x)|\;\le\;C\;\omega_1\left(f,\sqrt{\gamma(n,x,\alpha,z)}\;\right),$$

which completes the proof of the theorem.

Proof of Theorem 2. For the proof of *a*) see [1, Theorem A]. The proof of *b*) is a standard one [3, Chapter 9]: using (11), (8), [3, p. 141, (9.6.1)] and (9), we obtain for $g \in C^2[0,1]$:

$$\begin{aligned} |U_n^\alpha(g,x) - g(x)| &\le U_n^\alpha\left(\Big| \int_x^u \frac{|u-v|}{\varphi^2(v)} \cdot |\varphi^2(v) g''(v)|\, dv \ \Big|, x\right) \\ &\le \left(\frac{2}{n+1} + \alpha\right) \cdot \frac{1}{1+\alpha} \cdot \|\varphi^2 g''\|. \end{aligned}$$

Hence, by (10), we have

$$\begin{aligned} |U_n^\alpha(f,x) - f(x)| &\le |U_n^\alpha(f-g,x) - (f-g)(x)| + |U_n^\alpha(g,x) - g(x)| \\ &\le 2\, \|f-g\| + \left(\frac{2}{n+1} + \alpha\right) \cdot \frac{1}{1+\alpha} \cdot \|\varphi^2 g''\|. \end{aligned}$$

Using [3, p. 11, Theorem 2.1.1] we obtain

$$\|U_n^\alpha(f) - f\| \ \le\ C\, \omega_2^\varphi\left(f, \sqrt{\frac{1}{1+\alpha}\left(\frac{2}{n+1} + \alpha\right)}\ \right).$$

For *c*) we can write:

$$\|\bar{B}_n(f) - f\| \ \le\ \|\bar{B}_n(f) - S_n^{\frac{1}{n}}(f)\| + \|S_n^{\frac{1}{n}}(f) - f\|.$$

On the other hand, by (3), (5), (2) and *a*), we have
$|\bar{B}_n(f,x) - S_n^{\frac{1}{n}}(f,x)| \le$

$$\begin{aligned} &\le \frac{1}{\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}\, dt} \cdot \int_0^1 |f(t) - B_n(f,t)| \cdot t^{nx-1}(1-t)^{n(1-x)-1}\, dt \\ &\le \|f - B_n(f)\| \ \le\ C\, \omega_2^\varphi\left(f, \sqrt{\frac{1}{n}}\right) \end{aligned}$$

and

$$\|S_n^{\frac{1}{n}}(f) - f\| \ \le\ C\, \omega_2^\varphi\left(f, \sqrt{\frac{2}{n+1}}\ \right).$$

In conclusion

$$\|\bar{B}_n(f) - f\| \ \le\ C\ \left\{\omega_2^\varphi\left(f, \sqrt{\frac{1}{n}}\right) + \omega_2^\varphi\left(f, \sqrt{\frac{2}{n+1}}\ \right)\right\}.$$

For the proof of *d*) and *e*) we use

$$|f(u) - f(x)| \le \omega_1(f, |u-x|) \le \left(1 + \delta^{-4}(u-x)^2\right) \omega_1(f, \delta^2),$$

where $u, x \in [0,1]$ and $\delta > 0$. But, in view of [3, p. 25, Corollary 3.1.3] we have

$$\omega_1(f,\delta^2) \;\le\; C\ \omega_1^{\varphi}(f,\delta),$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. So

$$|f(u) - f(x)| \;\le\; C\ \left(1 + \delta^{-4}(u-x)^2\right)\omega_1^{\varphi}(f,\delta).$$

Hence

$$|F_n^*(f,x) - f(x)| \;\le\; C\ \left[1 + \delta^{-4}F_n^*((u-x)^2,x)\right]\omega_1^{\varphi}(f,\delta) \tag{20}$$

and

$$|\mathcal{F}_n^*(f,x) - f(x)| \;\le\; C\ \left[1 + \delta^{-4}\mathcal{F}_n^*((u-x)^2,x)\right]\omega_1^{\varphi}(f,\delta). \tag{21}$$

By (18), (19) and $\beta(n,x,\alpha,b,z) \le \beta'(n,\alpha,b,z)$, $\gamma(n,x,\alpha,z) \le \gamma'(n,\alpha,z)$ for all $x \in [0,1]$, we obtain

$$F_n^*((u-x)^2,x) \;\le\; \beta'(n,\alpha,b,z)$$

and

$$\mathcal{F}_n^*((u-x)^2,x) \;\le\; \gamma'(n,\alpha,z).$$

In conclusion, by (20) and choosing $\delta^4 = \beta'(n,\alpha,b,z)$ we get for $C = 2$ the assertion $d)$ of Theorem 2, and, by (21) and $\delta^4 = \gamma'(n,\alpha,z)$ we obtain for $C = 2$ the assertion $e)$ of Theorem 2.

For $f)$ we use again the standard method [3, Chapter 9]: if $g \in C_B[0,\infty)$ is twice differentiable such that $g', \varphi^2 g'' \in C_B[0,\infty)$ then, by [3, p. 141, (9.6.1)] and [7, (2.50)] for $b = c$, we have

$$\begin{aligned} |\tilde{L}_n(g,x) - g(x)| \;&\le\; \tilde{L}_n\left(\Big|\int_x^u \frac{|u-v|}{v}\cdot|vg''(v)|\,dv\,\Big|, x\right) \\ &\le\; \tilde{L}_n\left(\frac{(u-x)^2}{x}, x\right)\cdot\|\varphi^2 g''\|_\infty = \frac{2}{n}\cdot\|\varphi^2 g''\|_\infty. \end{aligned}$$

Because $\tilde{L}_n(1,x) = 1$ (see [7, (2.50)]) we get $\|\tilde{L}_n(f)\|_\infty \le \|f\|_\infty$, $f \in C_B[0,\infty)$. Thus

$$\begin{aligned} \|\tilde{L}_n(f) - f\|_\infty \;&\le\; \|\tilde{L}_n(f-g) - (f-g)\|_\infty + \|\tilde{L}_n(g) - g\|_\infty \\ &\le\; 2\,\|f-g\|_\infty + \frac{2}{n}\,\|\varphi^2 g''\|_\infty. \end{aligned}$$

Hence

$$\|\tilde{L}_n(f) - f\|_\infty \;\le\; 2\ K_2^{\varphi}\left(f, \frac{1}{n}\right)_\infty \;\le\; C\ \omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right)_\infty$$

(see [3, p. 11, Theorem 2.1.1] for the equivalence between $K_2^{\varphi}\left(f, \frac{1}{n}\right)_\infty$ and $\omega_2^{\varphi}\left(f, \sqrt{\frac{1}{n}}\right)_\infty$).

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