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AN EXISTENCE UNIQUENESS THEOREM FOR AN INTEGRAL EQUATION MODELLING INFECTIOUS DISEASES

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Abstract. By using a global inversion theorem due to R. Plastock [3], we prove an existence uniqueness result concerning the initial-value problem for the delay nonlinear integral equation $x(t) = \psi(t) + \int_{t-\tau}^{t} f(s, x(s))ds$. We establish also the continuous dependence on ψ of the solution of this equation.

1. **Introduction**

To describe the spread of certain infectious diseases, K. L. Cooke and J. L. Kaplan [1] proposed the following delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds. \tag{1}$$

In this equation, $x(t)$ is the proportion of infectives in a population at time t, τ is the length of time an individual remains infectious, and $f(t, x(t))$ is the proportion of new infectives per unit time.

It should be mentioned that Eq. (1) can be also interpreted as an evolution equation for a single species population. In this case, $x(t)$ is the number of individuals at time t, τ is the lifetime, and $f(t, x(t))$ is the number of new births per unit time. It is assumed that each individual lives exactly to the age τ, and then dies.

In this paper we are concerned with the initial-value problem associated to the equation

$$x(t) = \psi(t) + \int_{t-\tau}^{t} f(s, x(s))ds. \tag{2}$$

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More precisely, we look for positive continuous solutions of Eq. (2), when the proportion $\phi(t)$ of infectives is known for $t \in [-\tau, 0]$, i.e.

$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0]. \tag{3}$$

Obviously, we must assume that ϕ and ψ satisfy the equality

$$\phi(0) = \psi(0) + \int_{-\tau}^{0} f(s, \phi(s))ds. \tag{4}$$

Conditions ensuring the existence of at least one positive continuous solution of the initial-value problem (2)–(3) (with $\psi = 0$) have been given by R. Precup [4, 5, 6], E. Kirr [2], R. Precup and E. Kirr [7], and T. Trif [12]. It should be noted that, essentially, all these papers make use of different fixed point theorems. In the present paper we provide another approach of the problem (2)–(3). Namely, we obtain at once an existence uniqueness result, as well as the continuous dependence on ψ of the solution, for the initial-value problem (2)–(3) by using the following global inversion theorem due to R. Plastock [3] as the basic tool:

Theorem 1 ([3, Corollary 2.3]). *Let E and F be Banach spaces, let $A : E \to F$ be a local homeomorphism, and let $u : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function. Assume that the following conditions are satisfied:*

(i) *$u(\cdot, s)$ is strictly increasing for every $s > 0$ and $u(0, s) = 0$ for all $s \in \mathbf{R}_+$;*

(ii) *$\lim_{\|x\| \to \infty} \|A(x)\| = \infty$;*

(iii) *there exists a completely continuous operator $A_1 : E \to F$ such that the operator $A_2 := A + A_1$ satisfies*

$$\|A_2(x) - A_2(y)\| \geq u(\|x - y\|, r)$$

for every $r > 0$ and all $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$.

Then A is a (global) homeomorphism.

2. **Main result**

Concerning the initial-value problem (2)–(3) we will use the following hypotheses:

(H$_1$) $f : [-\tau, \infty[\, \times \mathbf{R} \to \mathbf{R}$ is a continuous function whose partial derivative with respect to the second argument, denoted by $f'_x(t,x)$, is continuous on $[-\tau, \infty[\, \times \mathbf{R}$;

(H$_2$) a is a positive real number, while $\phi : [-\tau, 0] \to [a, \infty[$ and $\psi : [0, \infty[\, \to \mathbf{R}$ are continuous functions satisfying (4);

(H$_3$) there exists a locally integrable function $b : [-\tau, \infty[\, \to \mathbf{R}$ such that

$$f(t,u) \geq b(t) \qquad \text{for all } (t,u) \in [-\tau, \infty[\, \times [a, \infty[$$

and

$$\psi(t) + \int_{t-\tau}^{t} b(s)ds > a \qquad \text{for all } t \in \mathbf{R}_+;$$

(H$_4$) there exist a continuous function $g : \mathbf{R}_+ \to \mathbf{R}_+$ and a continuous nondecreasing function $h : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\int_1^\infty \frac{1}{h(u)}\, du = \infty, \tag{5}$$

such that

$$|f(t,u)| \leq g(t)h(|u|) \qquad \text{for all } (t,u) \in \mathbf{R}_+ \times \mathbf{R}.$$

Theorem 2. *Suppose that the hypotheses* (H$_1$)–(H$_4$) *are fulfilled. Let T be an arbitrary positive real number and let E be the Banach space consisting of all continuous functions from $[-\tau, T]$ to $\mathbf{R}$, endowed with the usual sup-norm. Then the operator $A : E \to E$, defined by*

$$\begin{aligned} A(x)(t) &:= x(t) - \phi(t) + \psi(0) && \text{if } t \in [-\tau, 0] \\ A(x)(t) &:= x(t) - \textstyle\int_{t-\tau}^{t} f(s, x_\phi(s))ds && \text{if } t \in\,]0, T] \end{aligned}$$

for all $x \in E$ and all $t \in [-\tau, T]$, where $x_\phi : [-\tau, T] \to \mathbf{R}$ is the function defined by

$$x_\phi(t) := \begin{cases} \phi(t) & \text{if } t \in [-\tau, 0] \\ x(t) & \text{if } t \in\,]0, T], \end{cases}$$

is a global homeomorphism. In particular, there exists a unique continuous function $x : [-\tau, T] \to [a, \infty[$, satisfying (3) *and* (2) *for all $t \in\,]0, T]$.*

Proof. It is immediately seen that A is correctly defined because the hypotheses (H$_1$) and (H$_2$) guarantee that $A(x)$ is a continuous function for each $x \in E$.

Further, let $A_1 : E \to E$ be the operator defined by $A_1(x) := Id_E(x) - A(x)$, i.e.

$$A_1(x)(t) := \phi(t) - \psi(0) \qquad \text{if } t \in [-\tau, 0]$$
$$A_1(x)(t) := \int_{t-\tau}^{t} f(s, x_\phi(s))ds \qquad \text{if } t \in]0, T]$$

for all $x \in E$ and all $t \in [-\tau, T]$.

From the definitions of A and A_1 it follows that for all $x, y \in E$ we have

$$|A(x)(t) - A(y)(t)| = |x(t) - y(t)|$$
$$|A_1(x)(t) - A_1(y)(t)| = 0$$

if $t \in [-\tau, 0]$, whilst

$$|A(x)(t) - A(y)(t)| \leq |x(t) - y(t)| + \int_0^t |f(s, x(s)) - f(s, y(s))|ds$$
$$|A_1(x)(t) - A_1(y)(t)| \leq \int_0^t |f(s, x(s)) - f(s, y(s))|ds$$

if $t \in]0, T]$. These inequalities ensure that A and A_1 are continuous and that A_1 is completely continuous, by virtue of the Arzelá–Ascoli theorem.

Now we prove that A is a local homeomorphism. In fact, we will prove a little bit more: for all $x \in E$ and all $r > 0$, the restriction of A to the ball $B(x, r)$ is injective. To see this, let $x \in E$ and $r > 0$ be arbitrarily chosen. Further, let $y, z \in B(x, r)$ be so that $A(y) = A(z)$. Since $A(y)(t) = y(t) - \phi(t) + \psi(0)$ and $A(z)(t) = z(t) - \phi(t) + \psi(0)$ for all $t \in [-\tau, 0]$, it follows that $y(t) = z(t)$ for all $t \in [-\tau, 0]$. On the other hand, if we set $m_x := \min\, x([0, T])$, $M_x := \max\, x([0, T])$, and

$$M := \max\{\, |f'_x(s, u)| \mid s \in [0, T],\ u \in [m_x - r, M_x + r]\,\},$$

then for each $t \in [0, T]$ it holds that

$$y(t) - \int_{t-\tau}^{t} f(s, y_\phi(s))ds = z(t) - \int_{t-\tau}^{t} f(s, z_\phi(s))ds,$$

hence

$$\begin{aligned} |y(t) - z(t)| &\leq \int_{t-\tau}^{t} |f(s, y_\phi(s)) - f(s, z_\phi(s))|ds \\ &\leq \int_0^t |f(s, y(s)) - f(s, z(s))|ds \\ &\leq M \int_0^t |y(s) - z(s)|ds. \end{aligned}$$

By the Gronwall inequality we conclude that $y(t) = z(t)$ for all $t \in [0, T]$. Hence $y = z$ and A is a local homeomorphism, as claimed.

Next, we prove that A satisfies condition (ii) in Theorem 1. To this end, remark that for each $x \in E$ and each $t \in [0, T]$ it holds that

$$x(t) = A(x)(t) + \int_{t-\tau}^{t} f(s, x_\phi(s))ds.$$

Taking into account the hypotheses (H_2) and (H_4), we deduce that

$$\begin{aligned} |x(t)| &\leq \|A(x)\| + \int_{-\tau}^{0} f(s, \phi(s))ds + \int_{0}^{t} |f(s, x(s))|ds \\ &\leq \|A(x)\| + \phi(0) + \int_{0}^{t} g(s)h(|x(s)|)ds. \end{aligned}$$

By a modified version of the Gronwall inequality (see M. Rădulescu and S. Rădulescu [9, p. 103]) we conclude that

$$\int_{\|A(x)\|+\phi(0)}^{|x(t)|} \frac{1}{h(u)}\, du \leq \int_{0}^{t} g(s)ds \leq \int_{0}^{T} g(s)ds$$

for all $x \in E$ and all $t \in [0, T]$. This inequality and (5) imply the validity of the condition (ii) in Theorem 1.

In conclusion, all the conditions in Theorem 1 are satisfied if the function $u : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is defined by $u(r, s) := r$. Therefore, A is a global homeomorphism, hence there exists a unique $x \in E$ such that $A(x) = \tilde{\psi}$, where $\tilde{\psi} : [-\tau, T] \to \mathbf{R}$ is the function defined by

$$\tilde{\psi}(t) := \begin{cases} \psi(0) & \text{if } t \in [-\tau, 0] \\ \psi(t) & \text{if } t \in]0, T]. \end{cases}$$

Clearly, x satisfies (3) and (2) for all $t \in]0, T]$. We claim that $x(t) \geq a$ for all $t \in [-\tau, T]$. To see this, set

$$T_0 := \inf \{ t \in [-\tau, T] \ \mid\ \forall\ s \in [-\tau, t] \ :\ x(s) \geq a \}.$$

According to (H_2), we have $T_0 \geq 0$. Assume that $T_0 < T$. Since the function

$$\forall\ t \in \mathbf{R}_+ \quad \longmapsto \quad \psi(t) + \int_{t-\tau}^{t} b(s)ds \in \mathbf{R}$$

is continuous, it follows from (H_3) that the real number ε, defined by

$$\varepsilon := \min_{t \in [0,T]} \left(\psi(t) + \int_{t-\tau}^{t} b(s)ds \right) - a,$$

is positive. Set $\alpha := \min x([-\tau, T])$, $\beta := \max x([-\tau, T])$,

$$\gamma := \max \{ |f(s,u)| \mid s \in [-\tau, T],\ u \in [\alpha, \beta] \},$$

and then choose $\delta > 0$ such that $4\gamma\delta \leq \varepsilon$ and

$$|\psi(t) - \psi(T_0)| < \frac{\varepsilon}{2} \qquad \text{for all } t \in [T_0, T_0 + \delta] \cap [0, T].$$

The assumption $T_0 < T$ implies the existence of a point $t \in [T_0, T_0 + \delta] \cap [0, T]$ such that $x(t) < a$. But, on the other hand, we have

$$\begin{aligned} x(t) &= \psi(t) + \int_{T_0-\tau}^{T_0} f(s, x(s))ds - \int_{T_0-\tau}^{t-\tau} f(s, x(s))ds + \int_{T_0}^{t} f(s, x(s))ds \\ &\geq \psi(T_0) + \int_{T_0-\tau}^{T_0} b(s)ds - \int_{T_0-\tau}^{t-\tau} |f(s, x(s))|ds - \int_{T_0}^{t} |f(s, x(s))|ds \\ &\qquad + \psi(t) - \psi(T_0) \\ &\geq a + \varepsilon - 2\gamma(t - T_0) - \frac{\varepsilon}{2} \geq a + \frac{\varepsilon}{2} - 2\gamma\delta \geq a. \end{aligned}$$

The obtained contradiction shows that $T_0 = T$. Consequently, $x(t) \geq a$ for all $t \in [-\tau, T]$. $\square$

Suppose that the hypotheses (H_1)–(H_4) are satisfied. Let T and E be as in the statement of Theorem 2, and let $x : [-\tau, T] \to [a, \infty[$ be the unique continuous function satisfying (3) and (2) for all $t \in]0, T]$. Further, let $\psi_n : [0, \infty[\to \mathbf{R}$ ($n \in \mathbf{N}$) be a sequence of continuous functions satisfying

$$\psi_n(0) = \phi(0) - \int_{-\tau}^{0} f(s, \phi(s))ds$$

and

$$\psi_n(t) + \int_{t-\tau}^{t} b(s)ds > a \qquad \text{for all } t \in \mathbf{R}_+,$$

for each positive integer n. According to Theorem 2, for every n there exists a unique continuous function $x_n : [-\tau, T] \to [a, \infty[$ such that

$$\begin{aligned} &x_n(t) = \phi(t) && \text{for all } t \in [-\tau, 0] \\ &x_n(t) = \psi_n(t) + \textstyle\int_{t-\tau}^{t} f(s, x_n(s))ds && \text{for all } t \in]0, T]. \end{aligned}$$

Corollary 3. *If $(\psi_n) \to \psi$ uniformly on $[0, T]$, then $(x_n) \to x$ uniformly on $[-\tau, T]$.*

Proof. For each positive integer n, let $\tilde{\psi}_n : [-\tau, T] \to \mathbf{R}$ be the function defined by

$$\tilde{\psi}_n(t) := \begin{cases} \psi_n(0) & \text{if } t \in [-\tau, 0] \\ \psi_n(t) & \text{if } t \in]0, T]. \end{cases}$$

Then for all n we have $x_n = A^{-1}(\tilde{\psi}_n)$. On the other hand, $x = A^{-1}(\tilde{\psi})$, where $\tilde{\psi}$ is defined as in the proof of Theorem 2. Since A^{-1} is continuous and $(\tilde{\psi}_n) \to \tilde{\psi}$ uniformly on $[-\tau, T]$, we conclude that $(x_n) \to x$ uniformly on $[-\tau, T]$. □

Corollary 4. *Suppose that the hypotheses* (H_1)–(H_4) *are fulfilled. Then there exists a unique continuous function* $x : [-\tau, \infty[\ \to \ [a, \infty[$, *satisfying* (3) *and* (2) *for all* $t \in]0, \infty[$.

Proof. According to Theorem 2, for each $T > 0$ there exists a unique continuous function $x_T : [-\tau, T] \to [a, \infty[$, satisfying (3) and (2) for all $t \in]0, T]$. Therefore, for all $T_1 > T_2 > 0$ and all $t \in]0, T_2]$ it holds that $x_{T_1}(t) = x_{T_2}(t)$. This remark enables us to define the function $x : [-\tau, \infty[\ \to \ [a, \infty[$ as follows: given $t \in [-\tau, \infty[$, select a real number $T \geq t$ and then set $x(t) := x_T(t)$. Clearly, x is the unique continuous function from $[-\tau, \infty[$ to $[a, \infty[$, satisfying (3) and (2) for all $t \in]0, \infty[$. □

Example. Let λ be a real number satisfying

$$\lambda \min_{t \in [0, \pi]} \int_{t-2}^{t} \ln(1 + \sin^2 s) ds > 1 \tag{6}$$

and let γ_0 be a root of the equation

$$\sqrt{\gamma} = \lambda \int_0^2 \ln(1 + \gamma^2 \sin^2 s) ds, \tag{7}$$

lying in $]1, \infty[$ (due to (6), Eq. (7) has at least one root in $]1, \infty[$). Then there exists a unique continuous function $x : [-2, \infty[\ \to \ [1, \infty[$, satisfying

$$\begin{array}{ll} x(t) = \gamma_0 & \text{for all } t \in [-2, 0] \\ x(t) = \int_{t-2}^{t} (\lambda + s_+) \sqrt{x(s)} \ln\left(1 + x^2(s) \sin^2 s\right) ds & \text{for all } t \in]0, \infty[, \end{array}$$

where $s_+ := \max\{0, s\}$.

This follows by Corollary 4 because all the hypotheses (H_1)–(H_4) are fulfilled if we choose $\tau := 2$, $a := 1$,

$$\begin{array}{ll} f : [-2,\infty[\, \times \mathbf{R} \to \mathbf{R} & f(t,u) := (\lambda + t_+)\sqrt{|u|}\ln(1+u^2\sin^2 t), \\ \phi : [-2,0] \to \mathbf{R} & \phi(t) := \gamma_0, \\ \psi : [0,\infty[\, \to \mathbf{R} & \psi(t) := 0, \\ b : [-2,\infty[\, \to \mathbf{R} & b(t) := \lambda \ln(1+\sin^2 t), \\ g : \mathbf{R}_+ \to \mathbf{R}_+ & g(t) := \lambda + t, \\ h : \mathbf{R}_+ \to \mathbf{R}_+ & h(u) := \sqrt{u}\ln(1+u^2). \end{array}$$

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