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CATEGORICAL SEQUENCES AND APPLICATIONS

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Abstract. Ralph Fox characterized the Lusternik-Schnirelmann category using the categorical sequences. In this paper we define the notion of G-categorical sequence, where G is a compact Lie group, and we prove that the result of Fox remains true for the equivariant Lusternik-Schnirelmann category.

1. **Introduction**

In the study of some problems of differential geometry, L. Lusternik and L. Schnirelmann introduced a new numerical topological invariant, defined for every closed subset A of a manifold M, called the category (Lusternik- Schnirelmann category) of A in M. This number is the minimum cardinality of a categorical covering of A in M, where "categorical covering" means a covering by categorical sets (see [5]).

This is a well-known and much studied homotopy invariant (see [3],[4],[5]). It gives important informations about the existence of critical points: when M is a smooth manifold, the Lusternik- Schnirelmann category of M is a lower bound for the number of critical points of a smooth function on M.

2. **Categorical sequences**

Let M be a topological space. A subset $A \subset M$ is called categorical in M if there exists an open subset $U \subset M$ such that $A \subset U$ and U is contractible in M.

Following Fox [3], we define the category of $X \subset M$ in M by the minimal number k such that X can be covered by k categorical subsets in M. We denote $cat(X, M) = k$.

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Because every subset of a contractible set in M is also contractible in M, we obtain that every categorical subset of M is contractible in M; the converse is not true.

A covering of X by categorical subsets of M is called a categorical covering of X in M; a categorical covering which verifies the minimal condition from definition is called minimal categorical covering.

Definition 2.1. A finite sequence $\{A_1, A_2, \ldots, A_k = X\}$ of closed subsets of X is called a categorical sequence of X in M if:

(i) $A_1 \subset A_2 \subset \ldots \subset A_k$

(ii) $A_1, A_2 - A_1, \ldots, A_k - A_{k-1}$ are categorial subsets in M.

The number k is called the length of categorical sequence $\{A_1, A_2, \ldots, A_k\}$.

Ralph Fox [3] established the following characterization of category in terms of categorical sequences:

Theorem 2.1. *Let M be a separable, arcwise connected, metric space, and let $X \subset M$ be a subspace such that $cat(X, M) < \infty$.*

Then the category of X in M, $cat(X, M)$, is the minimum of the lengths of the categorical sequences of X in M.

3. Categorical sequences method for equivariant Lusternik-Schnirelmann category

For the definition and the properties of equivariant category we follow Fadell [2].

Let M be a topological space and let G be a compact Lie group which acts on M. Let A be an invariant subspace of M. A homotopy $H : A \times I \longrightarrow M$ is called equivariant if $H(gx, t) = gH(x, t), \forall x \in A, \forall g \in G$.

Definition 3.1. The set A is called G-categorical in M if there exists an equivariant homotopy $H : A \times I \longrightarrow M$ such that $H_0 = H(\cdot, 0)$ is the inclusion and $H_1 = H(\cdot, 1)$ has the image in a single orbit $Orb(x)$.

Here $Orb(x) = \{gx | g \in G\} = Gx$ is the orbit of the point x. The G-orbits should be considered as "equivariant points". (A is G-categorical if it can be deformed equivariant in an orbit Gx.)

Definition 3.2. Let X be an invariant subspace of M. We say that X has G-category k in M and we denote $Gcat(X,M) = k$ if X can be covered by k G-categorical open subsets in M, and k is the minimal number with this property. If X cannot be covered by a finite number of such G-categorical open subsets in M, we say that $Gcat(X,M) = \infty$.

We define $Gcat(X,M) = 0$ if $X = \emptyset$.

If G acts trivially on M, then the G-category is exactly the Lusternik-Schnirelmann category.

In general, $Gcat(X,M) \geq cat(X/G, M/G)$. If the action of G on X is free, then $Gcat(X,M) = cat(X/G, M/G)$.

For G-category we know some properties, which are contained in the following proposition (see Fadell [2]):

Proposition 3.1. *(i) (normalisation) If X is an invariant subspace of M, G-categorical (open or closed), then*

$$Gcat(X,M) = 1$$

(ii) (monotonicity)If X, Y are two invariant subspaces of M and $X \subseteq Y$, then

$$Gcat(X,M) \leq Gcat(Y,M)$$

(iii) (subadditivity) If X, Y are two invariant subspaces of M, then

$$Gcat(X \cup Y, M) \leq Gcat(X,M) + Gcat(Y,M)$$

(iv) (invariance) If $\phi : M \longrightarrow M$ is an equivariant homeomorphism and X is an invariant subspace of M, then

$$Gcat(X,M) = Gcat(\phi(X), M)$$

(v) (continuity) If M is a G-ANR and X is an invariant subspace of M, then there is an open, invariant subset $U \subseteq M$ such that $X \subseteq U$ and

$$Gcat(X,M) = Gcat(U,M)$$

(vi) If $Gcat(X,M) = k$, then X has k orbits.

Now, we define the corresponding notion of categorical sequence in equivariant context:

Definition 3.3. We say that a finite sequence $\{A_1, A_2, \ldots, A_k = X\}$ of closed, invariant subsets of X is a G-categorical sequence of X in M if:

(i) $A_1 \subset A_2 \subset \ldots \subset A_k$

(ii) $A_1, A_2 - A_1, \ldots, A_k - A_{k-1}$ are G-categorical subsets in M.

k is called the length of G-categorical sequence $\{A_1, A_2, \ldots, A_k\}$.

The main result is contained in the following theorem:

Theorem 3.1. *Let M be a separable, arcwise connected, metric space and let G be a compact Lie group which acts on M. Let X be a invariant subspace of M such that $Gcat(X, M) < \infty$.*

In these conditions $Gcat(X, M)$ is the minimum of the lengths of the G-categorical sequences of X in M.

For the proof of this theorem, we need the following lemma:

Lemma 3.1. *Let M be a separable, arcwise connected, metric space and let G be a compact Lie group which acts on M. Let X and Y be two invariant subspaces of M such that X, Y are disjoint and open in their union $X \cup Y$.*

Then

$$Gcat(X \cup Y, M) = max\{Gcat(X, M), Gcat(Y, M)\}.$$

Proof. Let $X = \bigcup_{i \in I} X_i, Y = \bigcup_{i \in I} Y_j$, where the open subsets X_i and Y_j are G-categorical in M and these coverings of X and Y are minimal.

The covering $\{X_i \cup Y_j\}_{(i,j) \in I \times J}$ is open and G-categorical for $X \cup Y$ in M;it contains a subcovering by s sets such that $s = max\{\mid I \mid, \mid J \mid\}$. Then $Gcat(X \cup Y, M) \geq max\{Gcat(X, M), Gcat(Y, M)\}$.

From Proposition 3.1.(ii) we obtain $Gcat(X, M) \leq Gcat(X \cup Y, M)$ and $Gcat(Y, M) \leq Gcat(X \cup Y, M)$.

Then the above inequality holds.□

The proof of Theorem 3.1. We follow the method established by Fox in [3].

First, we will prove that if $\{A_1, A_2, \ldots, A_k = X\}$ is a G-categorical sequence of X in M, then $Gcat(X, M) \leq k$.

If $k = 1$, then $Gcat(X, M) \leq 1$.

Suppose this statement true for $k \leq r-1$; let $\{A_1, A_2, \ldots, A_k = X\}$ be a G-categorical sequence for X in M. Because A is G-categorical in M, by Proposition 3.1(v) there is an open, invariant subset $U \subset M$ such that $A_1 \subset U$ and U is G-categorical in M.

We prove that $\{A_2 - U, A_3 - U, \ldots, A_r - U\}$ is a G-categorical sequence of $X - U$ in M (with the length $r-1$). The sequence $\{A_1, A_2, \ldots, A_k = X\}$ is G-categorical; then $A_1 \subset A_2 \subset \ldots \subset A_k = X$ and $A_2 - U \subset A_3 - U \subset \ldots \subset A_r - U = X - U$. The set $A_2 - A_1$ is G-categorical in M and $A_2 - A_1 \subset A_2 - U$; then the set $A_2 - A_1$ is G-categorical in M. A_2 and U being invariant sets, we prove easily that $A_2 - U$ is invariant:

$\forall x \in A_2 - U, \forall g \in G \Leftrightarrow x \in A_2$ and $x \notin U$ and $g \in G \Leftrightarrow (x \in A_2$ and $g \in G)$ and $(x \notin U$ and $g \in G)$

We know that $gx \in A_2$; suppose that $gx \in U$. But U is invariant, so $g^{-1}gx \in U$ and we obtain that $x \in U$; this statement is a contradiction.

In the same way, we show that all the sets $A_k - U$ are invariants, for $k = \overline{2, r}$.

Also, the sets $(A_3 - U) - (A_2 - U) = A_3 - A_2, \ldots, (A_r - U) - (A_{r-1} - U) = A_r - A_{r-1}$ are G-categorical in M.

We just must justify that all these sets are closed, but this is very easy: $\overline{A_k - U} = \overline{A_k \cap (CU)} = \overline{A_k} \cap \overline{CU} = A_k \cap (CU) = A_k - U, k = \overline{2, r}$.

We conclude that the sequence $\{A_2 - U, A_3 - U, \ldots, A_k - U = X - U\}$ is a G-categorical sequence of $X - U$ in M; from the induction hypothesis, we obtain:

$$Gcat(X - U, M) \leq r - 1.$$

By using the subadditivity property of Proposition 3.1, we obtain:

$$Gcat(X, M) \leq Gcat(X - U, M) + Gcat(U, M) \leq (r-1) + 1 = r$$

Now, we will prove that there is a G-categorical sequence of X in M, such that its length is $\leq Gcat(X, M)$.

For $Gcat(X, M) = 1$ this statement is true.

Suppose that this is true also for $Gcat(X, M) \leq r-1$ and let $\{B_1, B_2, \ldots, B_r\}$ be a minimal, G-categorical, open covering of X in M.

We define the sets:

$$C_i = \{x \in X | x \in B_j, \forall j \leq i, x \notin B_j, \forall j > i\}, i = \overline{1, r};$$

these sets are closed in X.

We consider the sets C_1 and $X - B_1$; they are closed and disjoint in the (metric, so) normal space X. Then there is an open subset $D_1 \subset X$ such that

$$C_1 \subset D_1$$

$$\overline{D_1} \cap (X - B_1) = \emptyset$$

We suppose that we have $j-1$ open subsets $D_1, D_2, \ldots, D_{j-1}$ of X such that for $i \leq j-1$ the following relations are true:

$$C_i - D_1 \cup D_2 \cup \ldots \cup D_{i-1} \subset D_i$$

$$\overline{D_i} \cap (X - B_i) = \emptyset$$

The subsets $X - B_j$ and $C_j - \bigcup_{i<j} D_i$ are closed in X and disjoint:

$$(X - B_j) \cap (C_j - \bigcup_{i<j} D_i) \subset (X - B_j) \cap (C_j - C_{j-1}) \subset (X - B_j) \cap B_j = \emptyset$$

Then there is $D_j \subset X$ open such that:

$$C_j - \bigcup_{i<j} D_i \subset D_j$$

$$\overline{D_j} \cap (X - B_j) = \emptyset$$

For the subsets $D_1, D_2, \ldots, D_r$ as above, the following relations are true:

$$\overline{D_1} - D_1 \subset B_1 - C_1 \subset B_2 \cup B_3 \cup \ldots \cup B_r$$

$$\bigcup_{i \leq r} (\overline{D_i} - D_i) \subset B_2 \cup B_3 \cup \ldots \cup B_r$$

We obtain that

$$Gcat(\bigcup_{i \leq r} (\overline{D_i} - D_i), M) \leq r - 1$$

From the induction hypothesis, there is a G-categorical sequence $\{A_1, A_2, \ldots, A_{k-1} = \bigcup_{i \leq r} (\overline{D_i} - D_i)\}$ of the set $\bigcup_{i \leq r} (\overline{D_i} - D_i)$ in M, and its length is $k - 1 \leq r - 1$.

We prove that $\{X \cap A_1, X \cap A_2, \ldots, X \cap A_{k-1}, X\}$ is a G-categorical sequence of X in M.

All these sets are closed in X. From $A_1 \subset A_2 \subset \ldots \subset A_{k-1}$ we obtain that $X \cap A_1 \subset X \cap A_2 \subset \ldots \subset X \cap A_{k-1} \subset X$. The subsets $A_1, A_2, \ldots, A_{k-1}$ are G-invariant, so $X \cap A_1, X \cap A_2, \ldots, X \cap A_{k-1}$ are G-invariant. The subset A_1 is G-categorical in M and $X \cap A_1$ will be also G-categorical in M. The subsets $X \cap A_2 - X \cap A_1 = X \cap (A_2 - A_1), \ldots, X \cap A_{k-1} - X \cap A_{k-2} = X \cap (A_{k-1} - A_{k-2})$ are G-categorical in M, because $A_2 - A_1, \ldots, A_{k-1} - A_{k-2}$ are G-categorical in M.

We just must justify that $X - X \cap A_{k-1}$ is a G-categorical subset in M. It is easy to see that $X - X \cap A_{k-1} = X - X \cap (\cup_{i \leq r} (\overline{D_i} - D_i))$ is open in X ($\cup_{i \leq r} (\overline{D_i} - D_i)$ is closed in X) and it is invariant. Every component of $X - X \cap A_{k-1}$ is contained in one of the sets $D_i \subset B_i$; every B_i is G-categorical in M. By using Proposition 3.1(ii) and Lemma 3.1 we obtain that $X - X \cap A_{k-1}$ is G-categorical in M. □

G-categorical sequences can be used for the proof of product inequality; for nonequivariant case, the reader can see [3] and [4].

For two G-spaces X, Y, we define the action of G on the product space $X \times Y$ by

$$G \times (X \times Y) \longrightarrow X \times Y$$

$$g(x, y) = (gx, gy).$$

Proposition 3.2. *Let X, Y two separable, arcwise connected, metric G-spaces. If X and Y are G-invariant, then*

$$Gcat(X \times Y) \leq Gcat(X) + Gcat(Y) - 1$$

Proof. Let $\{A_1, A_2, \ldots, A_m = X\}$ be a G-categorical sequence of X in X and let $\{B_1, B_2, \ldots, B_n = Y\}$ be a G-categorical sequence of Y in Y. We consider the sets

$$C_k = \bigcup_{i+j=k+1} A_i \times B_j.$$

All these sets are closed and G-invariant (because $A_i, 1 \leq i \leq m$, $B_j, 1 \leq j \leq n$ are G-invariant).

From $A_1 \subset \ldots \subset A_m = X$ and $B_1 \subset \ldots \subset B_n = Y$, we obtain that $C_1 \subset \ldots \subset C_{m+n-1} = X \times Y$. We only must show that $\{C_1, C_2 - C_1, \ldots, C_{m+n-1} - C_{m+n-2}\}$ are G-categorical in $X \times Y$.

A_1 is G-categorical in X; then there is an equivariant homotopy $H_A : A_1 \times I \longrightarrow X$ such that $H_{X,0} = H_X(\cdot, 0)$ is the inclusion and $H_{X,1} = H_X(\cdot, 1)$ has the image in a single orbit $Orb(x_{A_1})$. The same holds for B_1 and the equivariant homotopy $H_Y : B_1 \times I \longrightarrow Y$, with corresponding orbit $Orb(y_{B_1})$. Then

$$H : (A_1 \times B_1) \times I \longrightarrow X \times Y$$

defined by

$$H((x, y), t) = (H_X(x, t), H_Y(y, t))$$

is G-invariant: $H(g(x, y), t) = H((gx, gy), t) = (H_X(gx, t), H_Y(gy, t)) = (gH_X(x, t), gH_Y(y, t)) = gH((x, y), t), \forall (x, y) \in X \times Y, \forall g \in G$. Also, $H(\cdot, 0)$ is the inclusion and $H(\cdot, 1)$ has the image in a single orbit $Orb(x_{A_1}, y_{B_1})$. We conclude that C_1 is G-categorical in $X \times Y$.

Writing $C_{k+1} - C_k = \bigcup_{i+j=k+2} (A_i - A_{i-1}) \times (B_j - B_{j-1})$, $1 \leq k \leq m + n - 2$, ($A_0 = \emptyset$ and $B_0 = \emptyset$ for convenience), it is easy to see that $(A_i - A_{i-1}) \times (B_j - B_{j-1})$ is G-categorical in $X \times Y$ and the sets $(A_i - A_{i-1}) \times (B_j - B_{j-1})$, $(A_{i'} - A_{i'-1}) \times (B_{j'} - B_{j'-1})$, $i + j = i' + j'; i \neq i', j \neq j'$, satisfy the assumption of Lemma 3.1.

Then $C_{k+1} - C_k$ is G-categorical in $X \times Y$. $\square$

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