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CHARACTERIZATIONS OF INJECTIVE MULTIPLIERS ON PARTIALLY ORDERED SETS

ÁRPÁD SZÁZ AND JÓZSEF TÚRI

Abstract. An ordered pair $(\mathcal{D}, \mathcal{E})$ of subsets of a partially ordered set $\mathcal{A}$ is called a pairing in $\mathcal{A}$ if the meet $D \wedge E = \inf\{D, E\}$ exists for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$. Moreover, the set $\mathcal{D}$ is said to separate the points of $\mathcal{E}$ if for each $E_1, E_2 \in \mathcal{E}$ with $E_1 \neq E_2$ there exists $D \in \mathcal{D}$ such that $D \wedge E_1 \neq D \wedge E_2$.

A function F of $\mathcal{D}$ to $\mathcal{E}$ is called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}$. Moreover, the function F is called a multiplier if $F(D_1) \wedge D_2 = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$. If in particular $\mathcal{D}$ is a meet semilattice in $\mathcal{A}$, then the function F is a nonexpansive multiplier if and only if $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1, D_2 \in \mathcal{D}$.

After summarizing some basic properties of pairings, nonexpansive functions and multipliers, it is shown that if F is a multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then $\mathcal{E}$ separates the points of $\mathcal{D}$ if and only if F is injective and $\mathcal{D}$ separates the points of $\mathcal{E}$. Moreover, some sufficient conditions are given in order that a nonexpansive function and a multiplier of $\mathcal{D}$ to $\mathcal{E}$ be the identity function of $\mathcal{D}$.

The results obtained naturally extends and supplement some former statements of G. Szász, J. Szendrei, Á. Száz and G. Pataki on multipliers on semilattices and partially ordered sets. Moreover, they are also closely related to the works of several mathematicians on the extensions of semilattices and semigroups by the module theoretic methods of R. E. Johnson, Y. Utumi, G. D. Findlay and J. Lambek.

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1. Partially ordered sets

According to Birkhoff [2, p.1] a nonvoid set $\mathcal{A}$ together with a reflexive, transitive and antisymmetric relation $\leq$ is briefly called a poset. The use of the script letter is mainly motivated by the fact that each poset $\mathcal{A}$ is isomorphic to a family of sets partially ordered by set inclusion. The isomorphism is established by the mapping $A \mapsto]\,A\,]$, where $A \in \mathcal{A}$ and $]\,A\,] = \{\, B \in \mathcal{A} : \quad B \leq A \,\}$.

As usual, a poset $\mathcal{A}$ is called (1) totally ordered if for each $A\,, B \in \mathcal{A}$ either $A \leq B$ or $B \leq A$ holds, (2) well-ordered if each nonvoid subset of $\mathcal{A}$ has a minimum (least element). Moreover, a subset $\mathcal{D}$ of $\mathcal{A}$ is called (1) descending if $A \in \mathcal{A}$, $D \in \mathcal{D}$ and $A \leq D$ imply $A \in \mathcal{D}$, and (2) cofinal if for each $A \in \mathcal{A}$ there exists $D \in \mathcal{D}$ such that $A \leq D$.

The infimum (greatest lower bound) and the supremum (least upper bound) of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ will be understood in the usual sense. However, instead of $\inf \mathcal{D}$ and $\sup \mathcal{D}$, we shall use the lattice theoretic notations $\bigwedge \mathcal{D}$ and $\bigvee \mathcal{D}$, respectively. Thus, for instance $E = \bigwedge \mathcal{D}$ if and only if $E \in \mathcal{A}$ such that for each $A \in \mathcal{A}$ we have $A \leq E$ if and only if $A \leq D$ for all $D \in \mathcal{D}$.

However, in the sequel, we shall only need some very particular cases of the above definitions whenever, for $A\,, B \in \mathcal{A}$, we write $A \wedge B = \inf\{\, A\,, B\}$ and $A \vee B = \sup\{\, A\,, B\}$. Concerning the operation $\wedge$, we shall frequently use the next simple theorems which, in their present forms, are usually not included in the standard books on lattices.

Theorem 1.1. *If $\mathcal{A}$ is a poset and $A\,, B\,, C\,, D \in \mathcal{A}$, then*

(1) $A \leq B$ if and only if $A = A \wedge B$; and thus $A = A \wedge A$;

(2) $A \leq B$ and $C \leq D$ imply $A \wedge C \leq B \wedge D$ whenever $A \wedge C$ and $B \wedge D$ exist.

Theorem 1.2. *If $\mathcal{A}$ is a poset and $A\,, B\,, C \in \mathcal{A}$, then*

(1) $A \wedge B = B \wedge A$ whenever either $B \wedge A$ or $A \wedge B$ exist;

(2) $(A \wedge B\,) \wedge C = A \wedge (\,B \wedge C\,)$ whenever $A \wedge B$ and $B \wedge C$ and moreover either $(A \wedge B\,) \wedge C$ or $A \wedge (\,B \wedge C\,)$ exist.

Remark 1.3. A slightly weaker form of the assertion (2) can be found in Birkhoff [2, Theorem 1, p.8]. Moreover, a somewhat weaker form of the dual of this assertion can be found in Grätzer [7, Exercise 31, p.8].

Theorem 1.4. *If $\mathcal{A}$ is a poset and $\mathcal{D} \subset \mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ is descending;

(2) $A \in \mathcal{A}$ and $D \in \mathcal{D}$ imply $A \wedge D \in \mathcal{D}$ whenever $A \wedge D$ exists.

Remark 1.5. From the above theorems, by using the dual $\mathcal{A}(\geq)$ of the poset $\mathcal{A}(\leq)$, one can easily get some analogous theorems for the operation $\vee$ and the ascending subsets of $\mathcal{A}(\leq)$. However, in the sequel, we shall mainly need the operation $\wedge$. Therefore, we shall assume here some rather particular terminology.

A nonvoid subset $\mathcal{B}$ of poset $\mathcal{A}$ is called a semilattice in $\mathcal{A}$ if $D \wedge E$ exists in $\mathcal{A}$ and belongs to $\mathcal{B}$ for all $D, E \in \mathcal{B}$. Moreover, a nonvoid subset $\mathcal{D}$ of a semilattice $\mathcal{B}$ in a poset $\mathcal{A}$ is called an ideal of $\mathcal{B}$ if $D \wedge E$ is in $\mathcal{D}$ for all $D \in \mathcal{D}$ and $E \in \mathcal{B}$. Note that, by Theorem 1.5, $\mathcal{D}$ is an ideal of $\mathcal{B}$ if and only if $\mathcal{D}$ is descending subset of $\mathcal{B}$.

If $\mathcal{D}$ and $\mathcal{E}$ are subsets of a poset $\mathcal{A}$ such that $D \wedge E$ exists for all $D \in \mathcal{D}$ and $E \in \mathcal{E}$, then we write $\mathcal{D} \wedge \mathcal{E} = \{ D \wedge E : \quad D \in \mathcal{D}, \quad E \in \mathcal{E} \}$. Note that if $\mathcal{B}$ is a semilattice in a poset $\mathcal{A}$, then $\mathcal{B} = \mathcal{B} \wedge \mathcal{B}$. Moreover, if $\mathcal{D}$ and $\mathcal{E}$ are ideals of $\mathcal{B}$, then $\mathcal{D} = \mathcal{D} \wedge \mathcal{B}$ and $\mathcal{D} \cap \mathcal{E} = \mathcal{D} \wedge \mathcal{E}$. Therefore, the ideal $\mathcal{D} \cap \mathcal{E}$ inherits some useful properties of $\mathcal{D}$ and $\mathcal{E}$.

2. **Separating pairings in posets**

Definition 2.1. For every subset $\mathcal{D}$ of a poset $\mathcal{A}$, we define

$$\mathcal{D}^* = \left\{ A \in \mathcal{A} : \quad \forall \ D \in \mathcal{D} : \quad \exists \ A \wedge D \right\}.$$

Concerning the mapping $*$ of $\mathcal{P}(\mathcal{A})$ to itself, we can easily establish the following

Theorem 2.2. *If $\mathcal{D}$ and $\mathcal{E}$ are subsets of a poset $\mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{E} \subset \mathcal{D}^*$; *(2) $\mathcal{D} \subset \mathcal{E}^*$.*

Proof. Suppose that the assertion (1) holds and $D \in \mathcal{D}$. Then, $E \in \mathcal{D}^*$ for all $E \in \mathcal{E}$. Therefore, $D \wedge E = E \wedge D$ exists for all $E \in \mathcal{E}$. Consequently, $D \in \mathcal{E}^*$, and thus the assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can now be immediately established by interchanging the roles of $\mathcal{D}$ and $\mathcal{E}$ in the implication $(1) \Longrightarrow (2)$.

Remark 2.3. From the above theorem, by [29, Lemma 2.3], it follows that the mappings $*$ and $*$ establish a Galois connection between the posets $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$.

Therefore, as an immediate consequence of [29, Theorem 2.4], we can also state

Theorem 2.4. *If $\mathcal{A}$ is a poset, then*

(1) $\mathcal{D}^* = \mathcal{D}^{***}$ for all $\mathcal{D} \subset \mathcal{A}$;

(2) the composite mapping $**$ is a closure operation on $\mathcal{P}(\mathcal{A})$ such that $\mathcal{P}(\mathcal{A})^* = \mathcal{P}(\mathcal{A})^{**}$;

(3) the restriction of the mapping $*$ to $\mathcal{P}(\mathcal{A})^*$ is an inversion invariant injection of $\mathcal{P}(\mathcal{A})^*$ onto itself.

Hence, by [29, Theorem 1.9], it is clear that in particular we also have

Corollary 2.5. *If $\mathcal{A}$ is a poset, then $\mathcal{P}(\mathcal{A})^*$ is a complete poset.*

Definition 2.6. If $\mathcal{D}$ and $\mathcal{E}$ are nonvoid subsets of $\mathcal{A}$ such that $\mathcal{E} \subset \mathcal{D}^*$, then we say that the ordered pair $(\mathcal{D}, \mathcal{E})$ is a pairing in $\mathcal{A}$.

Our prime example for pairings is described in the following

Theorem 2.7. *If $\mathcal{A}$ is a poset with $\mathcal{A}^* \neq \emptyset$, then $\mathcal{A}^*$ is the largest subset of $\mathcal{A}$ such that $(\mathcal{A}^*, \mathcal{A})$ is a pairing in $\mathcal{A}$. Moreover, $\mathcal{A}^*$ is a semilattice in $\mathcal{A}$.*

Proof. The first statement is immediate from Definition 2.6 and Theorem 2.2. To prove the second statement, note that if $A, B \in \mathcal{A}^*$ and $C \in \mathcal{A}$, then by Definition 2.1 $A \wedge B$ and $A \wedge (B \wedge C)$ exist. Therefore, by Theorem 1.2, $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ also exists. Thus, again by Definition 2.1, $A \wedge B \in \mathcal{A}^*$.

Definition 2.8. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that for any $E_1, E_2 \in \mathcal{E}$, with $E_1 \neq E_2$, there exists $D \in \mathcal{D}$ such that $E_1 \wedge D \neq E_2 \wedge D$, then we say that $\mathcal{D}$ separates the points of $\mathcal{E}$.

Concerning the existence of separating pairings, we can only state the following generalization of [28, Theorem 2.9], and its immediate consequences.

Theorem 2.9. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *is a cofinal subset of* $\mathcal{E}$, *then* $\mathcal{D}$ *separates the points of* $\mathcal{E}$.

Corollary 2.10. *If* $\mathcal{A}$ *is a poset such that* $\mathcal{A}^*$ *is cofinal in* $\mathcal{A}$, *then* $\mathcal{A}^*$ *separates the points of* $\mathcal{A}$.

Corollary 2.11. *If* $\mathcal{A}$ *is a semilattice, then* $\mathcal{A}$ *separates the points of itself.*

In the sequel, we shall also need the following rather particular

Theorem 2.12. *Let* $(\mathcal{D}, \mathcal{E})$ *be a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D} \subset \mathcal{E}$. *Suppose that* $\mathcal{U} \subset \mathcal{D}$ *and* $\mathcal{V} \subset \mathcal{E}$ *such that* $\mathcal{U} \wedge \mathcal{D} \subset \mathcal{U}$ *and* $\mathcal{U} \wedge \mathcal{V} \subset \mathcal{V}$. *Moreover, suppose that* $\mathcal{U}$ *separates the points* $\mathcal{D}$ *and* $\mathcal{V}$ *separates the points of* $\mathcal{U}$. *Then* $\mathcal{V}$ *also separates the points of* $\mathcal{D}$.

Proof. Suppose that $D_1, D_2 \in \mathcal{D}$ such that $D_1 \neq D_2$. Then, since $\mathcal{U}$ separates the points of $\mathcal{D}$, there exists $U \in \mathcal{U}$ such that $D_1 \wedge U \neq D_2 \wedge U$. Moreover, since $\mathcal{U} \wedge \mathcal{D} \subset \mathcal{U}$, we also have $D_1 \wedge U, D_2 \wedge U \in \mathcal{U}$. Therefore, since $\mathcal{V}$ separates the points of $\mathcal{U}$, there exists $V \in \mathcal{V}$ such that $(D_1 \wedge U) \wedge V \neq (D_2 \wedge U) \wedge V$. Hence, by Theorem 1.2, it follows that $D_1 \wedge (U \wedge V) \neq D_2 \wedge (U \wedge V)$. Moreover, since $\mathcal{U} \wedge \mathcal{V} \subset \mathcal{V}$, we also have $U \wedge V \in \mathcal{V}$. Therefore, the required assertion is also true.

Corollary 2.13. *If* $\mathcal{A}$ *is a poset and* $\mathcal{D}$ *is an ideal of* $\mathcal{A}^*$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{A}^*$ *and* $\mathcal{A}$ *separates the points of* $\mathcal{D}$, *then* $\mathcal{A}$ *also separates the points of* $\mathcal{A}^*$.

3. **Nonexpansive functions on posets**

Definition 3.1. A function F of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ to $\mathcal{A}$ is called nonexpansive if $F(D) \leq D$ for all $D \in \mathcal{D}$.

Clearly, the identity function $\Delta_{\mathcal{D}}$ of $\mathcal{D}$ is nonexpansive. Moreover, to provide a less trivial example, we can also at once state

Example 3.2. If $\mathcal{T}$ is a subset of an upper complete poset $\mathcal{A}$, then the function $\circ$, defined by $A^\circ = \sup\{V \in \mathcal{T}: \quad V \leq A\}$ for all $A \in \mathcal{A}$, is nonexpansive. Note that, in particular, $\mathcal{A}$ may be the family of all subsets of a set X and $\mathcal{T}$ may be a topology on X.

Remark 3.3. To let the reader feel the importance of nonexpansive functions, it is also worth mentioning that if F is a nonexpansive function of a poset $\mathcal{A}$ to itself, then the minimal elements of $\mathcal{A}$ are fixed points of F. The dual statement has previously been stressed by Bronsted [4].

By the corresponding definitions, we evidently have the following two theorems.

Theorem 3.4. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the function* F', *defined by* $F'(D) = F(D) \wedge D$ *for all* $D \in \mathcal{D}$, *is nonexpansive. Moreover,* F *is nonexpansive if and only if* $F' = F$.

Corollary 3.5. *If* F *is a nonexpansive function of an ideal* $\mathcal{D}$ *of a semilattice* $\mathcal{A}$ *onto a subset* $\mathcal{E}$ *of* $\mathcal{A}$, *then* $\mathcal{E} \subset \mathcal{D}$.

Theorem 3.6. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the following assertions are equivalent:*

(1) F *is nonexpansive;*

(2) $F(D_1) = F(D_1) \wedge D_2$ *for all* $D_1 \in \mathcal{D}$ *and* $D_2 \in \mathcal{A}$ *with* $D_1 \leq D_2$.

Remark 3.7. In this respect, it is also worth noticing that a function F of a poset $\mathcal{D}$ to another $\mathcal{E}$ is nondecreasing if and only if $F(D_1) = F(D_1) \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$.

Therefore, in addition to Theorem 3.6, we may also naturally state the following theorem of [18].

Theorem 3.8. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a function of* $\mathcal{D}$ *to* $\mathcal{E}$, *then the following assertions are equivalent:*

(1) F *is nonexpansive and nondecreasing;*

(2) $F(D_1) \wedge D_2 = F(D_1) \wedge F(D_2)$ *for all* $D_1, D_2 \in \mathcal{D}$ *with* $D_1 \leq D_2$.

In contrast to the injective nondecreasing functions, the inverse of an injective nonexpansive function need not be nonexpansive. Namely, we have the following natural extension of an observation of Szász [22, p.449].

Theorem 3.9. *If* F *is an injective function of a subset* $\mathcal{D}$ *of a poset* $\mathcal{A}$ *to* $\mathcal{A}$ *such that both* F *and* F^{-1} *are nonexpensive, then* $F = \Delta_{\mathcal{D}}$.

Proof. Note that, in this case, we have $D = F^{-1}\big(F(D)\big) \leq F(D) \leq D$, and hence $F(D) = D$ for all $D \in \mathcal{D}$.

Analogously to [8, (4.43) Theorem], we can also prove the following

Theorem 3.10. *If* F *is an injective nonexpansive function of an ideal* $\mathcal{D}$ *of a well-ordered set* $\mathcal{A}$ *to* $\mathcal{A}$, *then* $F = \Delta_{\mathcal{D}}$.

Proof. If this is not the case, then by the well-orderedness of $\mathcal{A}$ there exists a smallest element D of $\mathcal{D}$ such that $F(D) \neq D$. Hence, by the nonexpansibility of F, it follows that $F(D) < D$. Moreover, by using Corollary 3.5 and the injectivity of F, we can see that $F(D) \in \mathcal{D}$ and $F\big(F(D)\big) \neq F(D)$. But, this contradicts the minimality of D.

Moreover, as a dual of [8, (4.43) Theorem], we can also state

Theorem 3.11. *If F is an injective nondecreasing function of a dually well-ordered set $\mathcal{A}$ to itself, then F is nonexpansive.*

Hence, by using Theorem 3.9, we can easily derive

Corollary 3.12. *If F is an injective nondecreasing function of a dually well-ordered set $\mathcal{A}$ onto itself, then $F = \Delta_{\mathcal{A}}$.*

Proof. In this case, F^{-1} is also an injective nondecreasing function of $\mathcal{A}$ onto itself. Therefore, by Theorem 3.11, not only F, but also F^{-1} is nonexpansive. Therefore, Theorem 3.9 can be applied.

Moreover, as an immediate consequence of Theorems 3.11 and 3.10, we can also state

Corollary 3.13. *If F is an injective nondecreasing function of a well-ordered and dually well-ordered set $\mathcal{A}$ to itself, then $F = \Delta_{\mathcal{A}}$.*

4. **Nonexpansive multipliers on posets**

Definition 4.1. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$, then a function F of $\mathcal{D}$ to $\mathcal{E}$ is called a multiplier if $F(D_1) \wedge D_2 = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$.

The above definition can be illustrated with the following examples of [18].

Example 4.2. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$, then the identity function $\Delta_{\mathcal{D}}$ of $\mathcal{D}$ is a nonexpansive multiplier.

Example 4.3. If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, then for each $A \in \mathcal{E}$ the function F, defined by $F(D) = A \wedge D$ for all $D \in \mathcal{D}$, is a nonexpansive multiplier.

Example 4.4. Let $\mathcal{A}$ be a distributive lattice [2, p.12] with a least element O and a greatest element X such that $X \neq O$. Choose $A \in \mathcal{A}$ such that $A \neq O$, and define $\mathcal{D} = \{ D \in \mathcal{A} : \quad A \wedge D = O \}$ and $F(D) = A \vee D$ for all $D \in \mathcal{D}$. Then

$\mathcal{D}$ is an ideal of $\mathcal{A}$ such that $\mathcal{D}$ does not separate the points of $\mathcal{A}$, and F is a nonextendable multiplier such that $D < F(D)$ for all $D \in \mathcal{D}$.

Remark 4.5. Moreover, it is also worth noticing that F is meet-preserving and $\mathcal{D} \cap F(\mathcal{D}) = \emptyset$.

The importance of nonexpansive multipliers is also apparent from the following theorems of [18].

Theorem 4.6. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1) \wedge D_2 = F(D_1) \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$.

Corollary 4.7. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a nonexpansive multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then $(\mathcal{E}, \mathcal{E})$ is also a pairing in $\mathcal{A}$.*

Theorem 4.8. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then each of the following assertions implies the subsequent one:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1) = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$;

(3) $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1 \in \mathcal{D}$ and $D_2 \in \mathcal{A}$ with $D_1 \wedge D_2 \in \mathcal{D}$.

Corollary 4.9. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ and F is a nonexpansive multiplier of $\mathcal{D}$ to $\mathcal{E}$, then F is nondecreasing.*

Theorem 4.10. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, and F is a function of $\mathcal{D}$ to $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is a nonexpansive multiplier;

(2) $F(D_1 \wedge D_2) = F(D_1) \wedge D_2$ for all $D_1, D_2 \in \mathcal{D}$;

(3) $F(D_1) = D_1 \wedge F(D_2)$ for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \leq D_2$.

Corollay 4.11. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ is a semilattice in $\mathcal{A}$, and F is a nonexpansive multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then F is meet-preserving and $\mathcal{E}$ is also a semilattice in $\mathcal{A}$.*

Corollay 4.12. *If F is a nonexpansive multiplier of an ideal $\mathcal{D}$ of a semilattice $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$, then F is idempotent and $\mathcal{E}$ is also an ideal of $\mathcal{A}$.*

Moreover, as some straightforward generalizations of [28, Theorems 6.2 and 6.3], we can also prove the following two theorems.

Theorem 4.13. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{E}$ *, and* F *is a multiplier of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $F'(\mathcal{D}) \subset \mathcal{E}$ *, then* F *is nonexpansive.*

Proof. If $D \in \mathcal{D}$, then by the above assumption on F' we have $F'(D) \in \mathcal{E}$. Hence, by the corresponding definitions and Theorem 1.2, it is clear that

$$F'(D) \wedge Q = \big(F(D) \wedge D\big) \wedge Q = Q \wedge \big(F(D) \wedge D\big) = \big(Q \wedge F(D)\big) \wedge D =$$

$$\big(F(Q) \wedge D\big) \wedge D = F(Q) \wedge (D \wedge D) = F(Q) \wedge D = Q \wedge F(D) = F(D) \wedge Q$$

for all $Q \in \mathcal{D}$. Hence, since $\mathcal{D}$ separates the points of $\mathcal{E}$, it follows that $F'(D) = F(D)$. Therefore, $F' = F$, and thus by Theorem 3.3 F is nonexpansive.

Theorem 4.14. *Let* $(\mathcal{D}, \mathcal{E})$ *be a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{E} \wedge \mathcal{D} \subset \mathcal{E}$. *Suppose that* F *is a multiplier of a subset* $\mathcal{D}_F$ *to* $\mathcal{E}$ *such that* $\mathcal{D}_F$ *separates the points of* $\mathcal{E}$. *Define*

$$F^{-} = \big\{ (D, E) \in \mathcal{D} \times \mathcal{E} : \quad \forall\ Q \in \mathcal{D}_F : \quad E \wedge Q = D \wedge F(Q) \big\}.$$

Then F^{-} *is the largest multiplier of a subset* $\mathcal{D}_{F^{-}}$ *of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $F \subset F^{-}$. *Moreover, if in particular* $\mathcal{D}$ *is a semilattice in* $\mathcal{A}$*, then* $\mathcal{D}_{F^{-}}$ *is already an ideal of* $\mathcal{D}$.

5. **Injective multipliers on posets**

The following theorem has mainly been suggested by Máté [14, Proposition 4]. For a generalization, see also [26, Theorem 2.3].

Theorem 5.1. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *and* F *is a multiplier of* $\mathcal{D}$ *onto* $\mathcal{E}$ *, then the following assertions are equivalent:*

(1) $\mathcal{E}$ *separates the points of* $\mathcal{D}$*;*

(2) F *is injective and* $\mathcal{D}$ *separates the points of* $\mathcal{E}$.

Proof. Suppose that the assertion (1) holds. Then, for any $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$, there exists $E \in \mathcal{E}$ such that $D_1 \wedge E \neq D_2 \wedge E$. Hence, by choosing $D \in \mathcal{D}$ such that $E = F(D)$, we can see that

$$F(D_1) \wedge D = D_1 \wedge F(D) = D_1 \wedge E \neq D_2 \wedge E = D_2 \wedge F(D) = F(D_2) \wedge D.$$

Therefore, $F(D_1) \neq F(D_2)$, and thus the first part of the assertion (2) also holds.

Moreover, since for any $E_1, E_2 \in \mathcal{E}$ with $E_1 \neq E_2$ there exist $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$ such that $E_1 = F(D_1)$ and $E_2 = F(D_2)$, it is clear that the second part of the assertion (2) is also true.

Suppose now that the assertion (2) holds. Then, by the first part of the assertion (2), for any $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$, we have $F(D_1) \neq F(D_2)$. Therefore, by the second part of the assertion (2), there exists $D \in \mathcal{D}$ such that $F(D_1) \wedge D \neq F(D_2) \wedge D$. Hence, by defining $E = F(D)$, we can see that $E \in \mathcal{E}$ such that

$$D_1 \wedge E = D_1 \wedge F(D) = F(D_1) \wedge D \neq F(D_2) \wedge D = D_2 \wedge F(D) = D_2 \wedge E.$$

Therefore, the assertion (1) also holds.

Now, as some immediate consequences of Theorem 5.1, we can also state

Corollary 5.2. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{E}$ separates the points of $\mathcal{D}$, then every multiplier F of $\mathcal{D}$ onto $\mathcal{E}$ is injective.*

Corollary 5.3. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that there exists an injective multiplier F of $\mathcal{D}$ onto $\mathcal{E}$, then the following assertions are equivalent:*

(1) $\mathcal{D}$ separates the points of $\mathcal{E}$; *(2) $\mathcal{E}$ separates the points of $\mathcal{D}$.*

Corollary 5.4. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{E}$, and F is a multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then the following assertions are equivalent:*

(1) F is injective; *(2) $\mathcal{E}$ separates the points of $\mathcal{D}$.*

Moreover, by using Theorems 5.1 and 2.12, we can also prove the following

Theorem 5.5. *Let $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$. Suppose that F is an injective multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ onto a subset $\mathcal{E}_F$ of $\mathcal{E}$ such that $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ and $F'(\mathcal{D}_F) \subset \mathcal{E}$, and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Then $\mathcal{E}_F$ separates the points of $\mathcal{D}$.*

Proof. In this case, by Theorem 5.1, $\mathcal{E}_F$ separates the points of $\mathcal{D}_F$. Moreover, by Theorem 4.13, F is nonexpansive. Therefore, if $E \in \mathcal{E}_F$, then by choosing $D \in \mathcal{D}_F$ such that $E = F(D)$ and using Theorem 4.8, we can see that $E \wedge Q = F(D) \wedge Q = F(D \wedge Q) \in \mathcal{E}_F$ for all $Q \in \mathcal{D}$. Thus, in particular, $\mathcal{E}_F \wedge \mathcal{D}_F \subset \mathcal{E}_F$ also holds. Hence, by Theorem 2.12, it is clear that the required assertion is also true.

By the above theorem, it is clear that in particular we also have

Corollary 5.6. *Let $\mathcal{A}$ be a poset and suppose that F is an injective multiplier of an ideal $\mathcal{D}$ of $\mathcal{A}^*$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{A}$. Then $\mathcal{E}$ separates the points of $\mathcal{A}^*$.*

Moreover, by using Theorem 5.5, we can see that in some particular cases the maximal extension F^- of an injective multiplier F is also injective.

Theorem 5.7. *Let $(\mathcal{D}, \mathcal{E})$ be a pairing in a poset $\mathcal{A}$ such that $\mathcal{D} \subset \mathcal{E}$ and $\mathcal{E} \wedge \mathcal{D} \subset \mathcal{E}$. Suppose that F is an injective multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ to $\mathcal{E}$ such that $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Then F^- is also injective.*

Proof. In this case, by Theorem 5.5, the range $\mathcal{E}_F$ of F separates the points of $\mathcal{D}$. Hence, since F^- is an extension of F, it is clear that the range $\mathcal{E}_{F^-}$ of F^- separates the points of the domain $\mathcal{D}_{F^-}$ of F^-. Therefore, by Theorem 5.1, the required assertion is also true.

Corollary 5.8. *Let $\mathcal{A}$ be a poset and suppose that F is an injective multiplier of an ideal $\mathcal{D}$ of $\mathcal{A}^*$ to $\mathcal{A}$ such that $\mathcal{D}$ separates the points of $\mathcal{A}$. Then F^- is also injective.*

6. **Some further results on injective multipliers**

A counterpart of the following theorem is attributed to Devinatz and Hirschman by Wang [31, p.1134].

Theorem 6.1. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$, and F is an injective multiplier of $\mathcal{D}$ onto $\mathcal{E}$, then F^{-1} is an injective multiplier of $\mathcal{E}$ onto $\mathcal{D}$.*

Proof. In this case, we evidently have

$$F^{-1}(E_1) \wedge E_2 = F^{-1}(E_1) \wedge F\left(F^{-1}(E_2)\right) =$$

$$F\left(F^{-1}(E_1)\right) \wedge F^{-1}(E_2) = E_1 \wedge F^{-1}(E_2)$$

for all $E_1, E_2 \in \mathcal{E}$.

Now, we are ready to prove the following counterpart of Theorem 3.9.

Theorem 6.2. *If $(\mathcal{D}, \mathcal{E})$ is a pairing in a poset $\mathcal{A}$ such that $\mathcal{E}$ separates the points of $\mathcal{D}$, and F is a multiplier of a subset $\mathcal{D}_F$ of $\mathcal{D}$ onto $\mathcal{E}$ such that $F'(\mathcal{D}_F) \subset \mathcal{D} \cap \mathcal{E}$, then $F = \Delta_{\mathcal{D}_F}$.*

Proof. In this case, by Theorem 5.1, F is injective and $\mathcal{D}_F$ separates the points of $\mathcal{E}$. Hence, by Theorem 6.1, it follows that F^{-1} is a multiplier of $\mathcal{E}$ onto $\mathcal{D}_F$. Moreover, by Theorem 4.13, it is clear that not only F, but also F^{-1} is nonexpansive. Namely, we have

$$\big(F^{-1}\big)'(E) = F^{-1}(E) \wedge E = F\big(F^{-1}(E)\big) \wedge F^{-1}(E) = F'\big(F^{-1}(E)\big) \in \mathcal{D}$$

for all $E \in \mathcal{E}$. Therefore, by Theorem 3.9, the required assertion is also true.

Hence, it is clear that in particular we also have

Corollary 6.3. *If $\mathcal{A}$ is a poset and F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}^*$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $F'(\mathcal{D}) \subset \mathcal{A}^* \cap \mathcal{E}$ and $\mathcal{E}$ separates the points of $\mathcal{A}^*$, then $F = \Delta_{\mathcal{D}}$.*

Corollary 6.4. *If F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that $F'(\mathcal{D}) \subset \mathcal{E}$ and $\mathcal{E}$ separates the points of $\mathcal{A}^*$, then $F = \Delta_{\mathcal{D}}$.*

Moreover, by using Corollary 6.3, we can also prove the following

Theorem 6.5. *If $\mathcal{A}$ is a poset and F is a multiplier of a subset $\mathcal{D}$ of $\mathcal{A}^*$ to $\mathcal{A}$ such that the sets $F^{-1}(\mathcal{A}^*)$ and $F\left(F^{-1}(\mathcal{A}^*)\right)$ separates the points of $\mathcal{A}$ and $\mathcal{A}^*$, respectively, then $F = \Delta_{\mathcal{D}}$.*

Proof. Define $\mathcal{D}_0 = F^{-1}(\mathcal{A}^*)$ and $\mathcal{E}_0 = F\left(F^{-1}(\mathcal{A}^*)\right)$, and denote by F_0 the restriction of F to $\mathcal{D}_0$. Then, by the corresponding definitions, it is clear that $\mathcal{D}_0 \subset \mathcal{D}$ and $\mathcal{E}_0 \subset \mathcal{A}^*$ $\big($in fact, $\mathcal{E}_0 = F(\mathcal{D}) \cap \mathcal{A}^*\big)$, and F_0 is a multiplier of $\mathcal{D}_0$ onto $\mathcal{E}_0$. Moreover, from Theorem 4.13 we can see F_0 is nonexpansive. Therefore, $F_0{}' = F_0$, and thus $F_0{}'(\mathcal{D}_0) = \mathcal{E}_0$. Hence, by using Corollary 6.3, we can infer that $F_0 = \Delta_{\mathcal{D}_0}$. On the other hand, from Theorem 4.14, we know that F_0 has a unique maximal extension F_0^-. Therefore, we necessarily have $F_0^- = \Delta_{\mathcal{A}^*}$, and thus the required assertion is also true.

Now, as an immediate consequence of this theorem, we can also state

Corollary 6.6. *If F is a multiplier of a subset $\mathcal{D}$ of a poset $\mathcal{A}$ onto a subset $\mathcal{E}$ of $\mathcal{A}$ such that both $\mathcal{D}$ and $\mathcal{E}$ separate the points of $\mathcal{A}$, then $F = \Delta_{\mathcal{D}}$.*

From Theorem 6.2, by Theorem 5.1, it is clear that the following theorem is also true.

Theorem 6.7. *If* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{E}$*, and* F *is an injective multiplier of* $\mathcal{D}$ *onto* $\mathcal{E}$ *such that* $F'(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{E}$*, then* $F = \Delta_{\mathcal{D}}$.

Moreover, by using Theorem 5.5 instead of Theorem 5.1, we can also prove the following

Theorem 6.8. *Let* $(\mathcal{D}, \mathcal{E})$ *is a pairing in a poset* $\mathcal{A}$ *such that* $\mathcal{D} \subset \mathcal{E}$. *Suppose that* F *is an injective multiplier of a subset* $\mathcal{D}_F$ *of* $\mathcal{D}$ *to* $\mathcal{E}$ *such that* $\mathcal{D}_F \wedge \mathcal{D} \subset \mathcal{D}_F$ *and* $F'(\mathcal{D}_F) \subset \mathcal{D} \cap \mathcal{E}$*, and* $\mathcal{D}_F$ *separates the points of* $\mathcal{E}$. *Then* $F = \Delta_{\mathcal{D}_F}$.

Proof. In this case, by Theorem 4.13, F is nonexpansive. On the other hand, by Theorem 6.1, F^{-1} is a multiplier of the range $\mathcal{E}_F$ of F onto $\mathcal{D}_F$. Moreover, by Theorem 5.5, $\mathcal{E}_F$ separates the points of $\mathcal{D}$. Therefore, again by Theorem 4.13, F^{-1} is also nonexpansive. Namely, we again have $\left(F^{-1}\right)'(E) \in \mathcal{D}$ for all $E \in \mathcal{E}_F$. Therefore, by Theorem 3.9, the required assertion is also true.

Hence, it is clear that in particular we also have

Corollary 6.9. *If* $\mathcal{A}$ *is a poset and* F *is an injective multiplier of an ideal* $\mathcal{D}$ *of* $\mathcal{A}^*$ *to* $\mathcal{A}$ *such that* $F'(\mathcal{D}) \subset \mathcal{A}^*$ *and* $\mathcal{D}$ *separates the points of* $\mathcal{A}$*, then* $F = \Delta_{\mathcal{D}}$.

Corollary 6.10. *If* F *is an injective multiplier of an ideal* $\mathcal{D}$ *of a semilattice* $\mathcal{A}$ *to* $\mathcal{A}$ *such that* $\mathcal{D}$ *separates the points of* $\mathcal{A}$*, then* $F = \Delta_{\mathcal{D}}$.

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Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: `szaz@math.klte.hu`

Institute of Mathematics and Computer Science, College of Nyiregyháza, H-4400 Nyiregyháza, Pf. 166, Hungary
E-mail address: `turij@zeus.nyf.hu`

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