STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVII**, Number 1, March 2002

ON SPECTRA OF SOME TENSORS OF SIX–DIMENSIONAL KÄHLERIAN SUBMANIFOLDS OF CAYLEY ALGEBRA

M.B. BANARU

Abstract. The spectra of the metric tensor, of the almost complex structure, of the fundamental form, of the Riemannian curvature tensor, of the Ricci tensor and of the Weyl tensor of six–dimensional Kählerian submanifolds of Cayley algebra are computed.

It is proved that six–dimensional Kählerian submanifolds of Cayley algebra are CRK–manifolds, i.e. their Weyl tensor of conformal curvature is J–invariant.

1. **Preliminaries**

We consider an almost Hermitian manifold, i.e. a $2n$–dimensional manifold M^{2n} with Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G–structure, where G is the unitary group $U(n)$ [1]. Its elements are the frames adapted to the structure (A–frames). They look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\widehat{1}}, \ldots, \varepsilon_{\widehat{n}}),$$

where $p \in M^{2n}$, ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\widehat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here $a = 1, \ldots, n$; $\widehat{a} = a + n$.

2000 *Mathematics Subject Classification.* 53C10, 58C05.

Key words and phrases. almost Hermitian manifold, Kählerian manifold, tensor spectrum, CRK–manifold.

Therefore, the matrices of the operator of the almost complex structure and of the Riemannian metric written in an A–frame look as follows, respectively:

$$(J^k_j) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \quad (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right); \tag{1}$$

where I_n is the identity matrix; $k, j = 1, \dots, n$.

We recall that the fundamental form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in an A–frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right) \tag{2}$$

It is expedient to consider the other tensors written in an A–frame. This corresponds to the problems of the study of almost Hermitian manifolds. We remark that the notion of the tensor spectrum was introduced by V.F. Kirichenko [1].

2. **Kählerian structure on** $M^6 \subset \mathbf{O}$

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it well–known [2], two non–isomorphic three fold vector cross products are defined on it by means of the relations

$$P_1(X, Y, Z) = -X(\overline{Y}Z) + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$

$$P_2(X, Y, Z) = -(X\overline{Y})Z + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{O}$, $X \to \overline{X}$ is the conjugation operator. Moreover, any other three fold vector cross product in the octave algebra is isomorphic to one of the above–mentioned.

If $M^6 \subset \mathbf{O}$ is a six–dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle \cdot, \cdot \rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [2]. The submanifold $M^6 \subset \mathbf{O}$ is called Kählerian, if the following condition is fulfilled

$$\nabla J = 0,$$

where ∇ is the Levi–Civita connection of the metric. The point $p \in M^6$ is called general [3], if

$$e_0 \notin T_p(M^6) \text{ and } T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general–type submanifold [3]. In what follows all submanifolds M^6 to be considered are assumed to be of general–type.

3. **Riemannian curvature tensor spectrum**

The tensor of the Riemannian curvature (or Riemann–Christoffel tensor) plays an important role in the geometry of manifolds. The outstanding American mathematician Alfred Gray noted [4] that the identities the Riemannian curvature tensor satisfies are very important for the study of almost Hermitian manifolds. Taking into account the properties of the symmetry and of the reality of this tensor as well as the Ricci identity [5], [6], it is sufficient to obtain four (out of sixteen) types of components, that determine completely its spectrum.

Now, let $M^6 \subset \mathbf{O}$ be a six–dimensional Kählerian submanifold of the octave algebra. In [7] the Cartan structure equations of Kählerian have been obtained:

$$d\omega^a = \omega^a_b \wedge \omega^b;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b;$$

$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - T^7_{\widehat{a}\widehat{h}} T^7_{bg} \omega_h \wedge \omega^g,$$

where $\{T^7_{kj}\}$ are the components of the configuration tensor of M^6 [8] (or the Euler curvature tensor [9]). Here and further $a, b, c, d, g, h = 1, 2, 3$; $\widehat{a} = a + 3$; $k, j, m, l = 1, 2, 3, 4, 5, 6$.

Taking into account the fact that the Cartan structure equations must look as follows:

$$d\omega^k = \omega^k_j \wedge \omega^j;$$

$$d\omega^k_j = \omega^k_m \wedge \omega^m_j + \frac{1}{2} R^k_{jml} \omega^m \wedge \omega^l,$$

we compute the spectrum of the Riemannian curvature tensor of six–dimensional Kählerian submanifolds of Cayley algebra. We get such values

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0, \tag{3}$$

$$R_{\hat{a}b\hat{c}d} = -2T^{7}_{\hat{a}\hat{c}}T^{7}_{bd}.$$

We remark that the condition

$$R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0 \tag{4}$$

is a criterion [10] for an arbitrary almost Hermitian $M^6 \subset \mathbf{O}$ to belong to the class of $R1$–manifolds (in A. Gray's terminology [4], or parakählerian manifolds [11], or f–spaces [12]). But, however, A. Gray has proved [4] that every Kählerian manifold is parakählerian. That's why (4) could be obtained from the above–mentioned result [10].

4. **Ricci tensor spectrum**

We recall that the Ricci tensor of a Riemannian manifold [5], [6] is determined by the relation

$$ric_{kj} = R^{l}_{kjl}.$$

In view of the reality of the Ricci tensor for determining of its spectrum it is sufficient to find the components ric_{ab} and $ric_{\hat{a}b}$. From (3) we get:

$$ric_{ab} = 0, \quad ric_{\hat{a}b} = -2T^{7}_{\hat{a}\hat{c}}T^{7}_{cb}.$$

Since the condition $ric_{ab} = 0$ is a criterion for an arbitrary almost Hermitian manifold to possess a J–invariant Ricci tensor [13], we have the following Theorem.

Theorem I. *Every six–dimensional Kählerian submanifold of Cayley algebra possesses a J–invariant Ricci tensor.*

5. **Weyl tensor spectrum**

Now, we give the values of Weyl tensor spectrum of six–dimensional Kählerian submanifolds of the octave algebra. This tensor is determined by

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2}(ric_{ik}g_{jl} + ric_{jl}g_{ik} - ric_{il}g_{jk} - ric_{jk}g_{il}) +$$

$$+\frac{\mathcal{K}}{(n-1)(n-2)}(g_{jk}g_{il}-g_{jl}g_{ik}),$$

where $\mathcal{K}$ is the scalar curvature of M^6 [6]. Like in the case of the Riemannian curvature tensor, proceeding from the properties of the Weyl tensor, it is sufficient to find the components

$$W_{abcd}, \quad W_{\hat{a}bcd}, \quad W_{\hat{a}\hat{b}cd}, \quad W_{\hat{a}b\hat{c}d},$$

that determine completely the Weyl tensor spectrum. We obtain such values

$$W_{abcd}=0, \quad W_{\hat{a}bcd}=0,$$

$$W_{\hat{a}\hat{b}cd}=-\frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hc}\delta^b_d+T^7_{\hat{b}\hat{h}}T^7_{hd}\delta^a_c-T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^b_c-T^7_{\hat{b}\hat{h}}T^7_{hc}\delta^a_d)+\frac{\mathcal{K}}{20}\delta^{ba}_{cd},$$

$$W_{\hat{a}b\hat{c}d}=-2T^7_{\hat{a}\hat{c}}T^7_{bd}+\frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^c_b+T^7_{\hat{c}\hat{h}}T^7_{hb}\delta^a_d)+\frac{\mathcal{K}}{20}\delta^c_b\delta^a_d.$$

As the condition

$$W_{\hat{a}bcd}=0$$

is a criterion for an arbitrary almost Hermitian manifold to belong to the CRK–class (or $cR3$–class [14]), we obtain the following Theorem.

Theorem II. *Every six–dimensional Kählerian submanifold of Cayley algebra is a CRK–manifold.*

6. Table of classical tensors of six-dimensional Kählerian submanifolds of Cayley algebra

Let us put together the obtained results. The spectra of the structure tensors and of the fundamental form are found from (1) and (2). We remark that all these data define more exactly the result [15] obtained on six–dimensional Hermitian submanifolds of Cayley algebra.

Tensor	Tensor spectrum
Almost complex structure	$J_b^a = i\delta_b^a, \quad J_b^{\hat{a}} = 0, \quad J_{\hat{b}}^a = 0, \quad J_{\hat{b}}^{\hat{a}} = -i\delta_a^b$
Riemannian metric	$g_{ab}, \quad g_{\hat{a}b} = \delta_b^a, \quad g_{a\hat{b}} = \delta_a^b, \quad g_{\hat{a}\hat{b}} = 0$
Fundamental form	$F_{ab}, \quad F_{\hat{a}b} = -i\delta_b^a, \quad F_{a\hat{b}} = i\delta_a^b, \quad F_{\hat{a}\hat{b}} = 0$
Riemannian curvature tensor	$R_{abcd} = 0, \quad R_{\hat{a}bcd} = 0, \quad R_{\hat{a}\hat{b}cd} = 0,$ $R_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}}T^7_{bd}$
Ricci tensor	$ric_{ab} = 0, \quad ric_{\hat{a}b} = -2T^7_{\hat{a}\hat{c}}T^7_{cb}$
Weyl tensor	$W_{abcd} = 0, \quad W_{\hat{a}bcd} = 0,$ $W_{\hat{a}\hat{b}cd} = -\frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hc}\delta^b_d + T^7_{\hat{b}\hat{h}}T^7_{hd}\delta^a_c - T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^b_c -$ $-T^7_{\hat{b}\hat{h}}T^7_{hc}\delta^a_d) + \frac{\mathcal{K}}{20}\delta^{ba}_{cd},$ $W_{\hat{a}b\hat{c}d} = -2T^7_{\hat{a}\hat{c}}T^7_{bd} + \frac{1}{2}(T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^c_b + T^7_{\hat{c}\hat{h}}T^7_{hb}\delta^a_d) +$ $+\frac{\mathcal{K}}{20}\delta^c_b\delta^a_d$

References

[1] Kirichenko V.F., *Methods of the generalized Hermitian geometry in the theory of almost contact metric manifolds*, In: Problems of Geometry, 18(1986), 25–71.
[2] Gray A., *Vector cross products on manifolds*, Trans. Amer. Math. Soc., 141(1969), 465–504.
[3] Kirichenko V.F., *On nearly–Kählerian structures induced by means of 3–vector cross products on six–dimensional submanifolds of Cayley algebra*, Vestnik MGU, 3(1973), 70–75.
[4] Gray A., *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J., 28(1976), N4, 601–612.
[5] Kobayashi S., Nomizu K., *Foundation of differential geometry*, Inter–science, New-York–London-Sydney, 1969.
[6] Rashevskii P.K., *Riemannian geometry and tensor analysis*, Nauka, Moscow, 1964.
[7] Kirichenko V.F., *Classification of Kählerian structures induced by means of 3-vector cross products on six–dimensional submanifolds of Cayley algebra*, Izvestia vuzov, Kazan, 8(1980), 32–38.
[8] Gray A., *Some examples of almost Hermitian manifolds*, Illinois J. Math., 10(1966), N2, 353–366.
[9] Cartan E., *Riemannian geometry in an orthogonal frame*, MGU, Moscow, 1960.
[10] Banaru M., *On parakählerianity of six-dimensional Hermitian submanifolds of Cayley algebra*, Webs & Quasigroups, 1995, 81–83.
[11] Rizza G.B., *Varieta parakahleriane*, Ann. Mat. Pura ed Appl., 98(1974), N4, 47–61.
[12] Sawaki S., Sekigawa K., *Almost Hermitian manifolds with constant holomorphic sectional curvature*, J. Diff. Geom., 9(1974), 123–134.

[13] Arsen'eva O.E., Kirichenko V.F., *Self–dual geometry of generalized Hermitian surfaces*, Mat. Sbornik, 189(1998), N1, 21–44.
[14] Ignatochkina L.A., Kirichenko V.F., *Conformally invariant properties of nearly–Kählerian manifolds*, Mat. Zametki, 5(1999), 59–63.
[15] Banaru M.B., *On tensor spectrum of six–dimensional Hermitian submanifolds of Cayley algebra*, Recent Problems in Field Theory, Kazan, 2000, 18–21.

Smolensk State Pedagogical University, Prjevalsky str., 4,
Smolensk, 214000, RUSSIA

Received: 06.03.2001