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ON SOME INEQUALITIES FOR THE ε-ENTROPY NUMBERS

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Abstract. We prove the inequalities:
$\sum_{n=1}^{k} \alpha_n \epsilon_n \left(S_1 + \ldots + S_r\right) \leq (2^r - 1)\ c \sum_{n=1}^{k} \alpha_n \left(\epsilon_n(S_1) + \ldots + \epsilon_n(S_r)\right)$
and

$$\sum_{n=1}^{k} \alpha_n\, \epsilon_n(S_1\ \ldots\ S_r) \leq (2^r - 1)\ c \sum_{n=1}^{k} \alpha_n \epsilon_n(S_1) \ldots \epsilon_n(S_r),$$

$k = 1, 2, \ldots,\ r \geq 2$, where $(\epsilon_n(S))$ is the sequence of ϵ - entropy numbers of the linear and bounded operator $S : X \to X\, (S \in L(X))$ and (α_n) is such that $1 = \alpha_1 \geq \ldots \geq 0$ and $\alpha_{n^r} \leq \frac{c}{n^{r-1}} \alpha_n,\ \forall n \in N$. X is a Banach space.

1. **Introduction**

Let X be a Banach space and let $T : X \to X$ be a linear and bounded operator $(T \in L(X))$. The ϵ - entropy numbers of the operator T are defined, [1],[2],[4],[6], as follows:

$$\epsilon_n(T) = \inf\{\sigma > 0\ :\ \exists y_1, \ldots y_n \in X \text{ s.t. } TU_X \subseteq \cup_{i=1}^{n}\{y_i + \sigma U_X\}\},\ n = 1, 2, \ldots,$$

where $U_X = \{x \in X : \| x \| \leq 1\}$.

It is well known [1],[4],[6] that: $\| T \| = \epsilon_1(T) \geq \epsilon_2(T) \geq \ldots \geq 0$ and $\epsilon_{n_1 n_2}(S + T) \leq \epsilon_{n_1}(S) + \epsilon_{n_2}(T), \epsilon_{n_1 n_2}(ST) \leq \epsilon_{n1}(S)\epsilon_{n_2}(T),\ n_1, n_2 = 1, 2, \ldots$.

In the papers [5],[6] are presented the inequalities:

$$\sum_{n=1}^{k} \frac{\epsilon_n(S + T)}{n} \leq 3 \sum_{n=1}^{k} \frac{\epsilon_n(S) + \epsilon_n(T)}{n} \qquad (a)$$

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$$\sum_{n=1}^{k} \frac{\epsilon_n(ST)}{n} \leq 3 \sum_{n=1}^{k} \frac{\epsilon_n(S) \cdot \epsilon_n(T)}{n}, \qquad k = 1, 2, \ldots \qquad (b)$$

By reiteration we obtain:

$$\sum_{n=1}^{k} \frac{\epsilon_n\left(S_1 + \ldots + S_r\right)}{n} \leq 3^{r-1} \sum_{n=1}^{k} \frac{\epsilon_n(S_1) + \ldots + \epsilon_n(S_r)}{n} \qquad (a')$$

and an analog inequality (b') for the product of r operators.

In this paper we prove, in a simple way, that the factor 3^{r-1} can be replaced by $(2^r - 1)$.

More generally, is [6], the sequence $\left(\frac{1}{n}\right)$ is replaced by (α_n), where $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ and $\alpha_{n^2} \leq \frac{c}{n}\alpha_n, \ \forall n \in N$

2. **Results**

Firstly we remark that, from the inequalities of ϵ-entropy numbers for the sum and product of two operators we obtain:

Proposition 1.1 *The ϵ- entropy numbers verify the following inequalities:*

$$\epsilon_{n^r}\left(S_1 + \ldots + S_r\right) \leq \epsilon_n\left(S_1\right) + \ldots + \epsilon_n\left(S_r\right) \qquad (1)$$

$$\epsilon_{n^r}\left(S_1 \ldots S_r\right) \leq \epsilon_n\left(S_1\right) \ldots \epsilon_n\left(S_r\right) \qquad (2)$$

Now we prove:

Theorem 1.2. *The ϵ-entropy numbers verify the inequalities:*

$$\sum_{n=1}^{k} \alpha_n\, \epsilon_n\left(S_1 + \ldots + S_r\right) \leq \left(2^r - 1\right)\, c \sum_{n=1}^{k} \alpha_n\left(\epsilon_n\left(S_1\right) + \ldots + \epsilon_n\left(S_r\right)\right) \qquad (3)$$

$$\sum_{n=1}^{k} \alpha_n \epsilon_n\left(S_1\ \ldots\ S_r\right) \leq \left(2^r - 1\right)\, c \sum_{n=1}^{k} \alpha_n \epsilon_n\left(S_1\right)\ \ldots\ \epsilon_n\left(S_r\right), \qquad (4)$$

where (α_n) is a sequence such that $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ and $\alpha_{n^r} \leq \frac{c}{n^{r-1}}\alpha_n, \ \forall n \in N; \ k = 1, 2, \ldots$

Proof. We prove only (4). Tthe proof for (3) is similar. By using the inequality (2) and the fact that the sequence $(\epsilon_n(S))$ is non increasing we obtain:

$$\sum_{n=1}^{k} \alpha_n \epsilon_n (S_1 \dots S_r) \leq \sum_{n=1}^{(k+1)^r-1} \alpha_n \epsilon_n (S_1 \dots S_r) =$$

$$= \sum_{n=1}^{k} \sum_{i=n^r}^{(n+1)^r-1} \alpha_i \epsilon_i (S_1 \dots S_r) \leq$$

$$\leq \sum_{n=1}^{k} \left[(n+1)^r - n^r \right] \alpha_{n^r} \epsilon_{n^r} (S_1 \dots S_r) \leq$$

$$\leq \sum_{n=1}^{k} \left(2^k - 1\right) n^{r-1} \frac{c}{n^{r-1}} \alpha_n \epsilon_{n^r} (S_1 \dots S_r) \leq$$

$$\leq (2^r - 1)\ c \sum_{n=1}^{k} \alpha_n \epsilon_n (S_1) \dots \epsilon_n (S_r).$$

The proof is fulfiled.

3. **Application**

Let l_∞ be the normed space of all bounded sequence, where

$$\| x \|_\infty = \sup_n |x_n|.$$

For all $x \in l_\infty$, $card(x) = card\{n \in \mathcal{N} \ : \ x_n \neq 0\}$. We denote by K the set of all sequences $x \in l_\infty$ such that $card(x) \leq n$ and $x_1 \geq x_2 \geq \dots \geq 0$.
A function $\phi : K \to R$ is called symmetric norming function, [3],[4],[6], if:

1. $\phi(x) > 0,$ for $x \in K,\ x \neq 0;$
2. $\phi(\alpha_n) = \alpha \phi(x),\ \alpha \geq 0, x \in K;$
3. $\phi(x+y) \leq \phi(x) + \phi(y);$
4. $\phi(1, 0, 0, \dots) = 1;$
5. If $x, y \in K$ are such that

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i,\ k = 1, 2, \dots$$

   then $\phi(x) \leq \phi(y)$.

Example of such functions are: $\phi_\infty :\ x \in K \to x_1,\ \phi_1 :\ x \in K \to \sum_1^n x_i$ and $\phi_\omega :\ x \in K \to \sum_{i=1}^{n} \frac{x_i}{i}$.

It is known, [3],[6],[7], that, for all symmetric norming function ϕ, the functions: $\phi_{(p)} : (x_i) \in K \to (\phi(x_i^p))^{\frac{1}{p}}$, $1 \le p < \infty$ and $\overline{\phi} : (x_i) \in K \to \phi(\{\alpha_i x_i\})$ are symmetric norming functions.

If $x \in l_\infty$ are such that $x_1 \ge x_2 \ge \ldots \ge 0$, we consider

$$\phi(x) = \lim_{n\to\infty} \phi(x_1, \ldots, x_n, 0, 0, \ldots).$$

In [4], [7], the classes $L^{(\epsilon)}_{\phi_{(p)}}(X)$ are considered, where $L^{(\epsilon)}_{\phi_{(p)}}(X) = \{T \in L(X) : \phi_{(p)}(\{\epsilon_n(T)\}) < \infty\}$, $1 \le p < \infty$. If ϕ is replaced by $\overline{\phi}$, from the inequality (a) and the Minkowski inequality (for $\phi_{(p)}$,[3],[4],[7]) in [5], [7] is proved that

$\| T \|^{(\epsilon)}_{\overline{\phi}_{(p)}} = \overline{\phi}_{(p)}(\epsilon_n(T)) = (\phi(\{\alpha_n \epsilon_n^p(T)\}))^{\frac{1}{p}}$ is a quasi-norm.

From the above inequality (a') and the properties (2) and (5) of the functions ϕ, it results that:

$$\| \sum_{n=1}^{r} S_n \|^{(\epsilon)}_{\overline{\phi}} \le 3^{r-1} \sum_{n=1}^{r} \| S_n \|^{(\epsilon)}_{\overline{\phi}},$$

but from the theorem 1.2 we obtain that the factor 3^{r-1} can be replaced by $(2^r - 1)$ if $\alpha_n = \frac{1}{n}$, $n = 1, 2, \ldots$. A similar result is also true for all sequences (α_n) as above.

Remarks: For the dyadic entropy numbers $e_n(T) = \epsilon_{2^{n-1}}(T)$, $n = 1, 2, \ldots$, are known, [4] , [7], the inequalities:

$$\sum_{n=1}^{k} e_n(S \star T) \le 2 \sum_{n=1}^{k} e_n(S) \star e_n(T),$$

where $\star$ is $+$ or $\bullet$.

For the case of r operators $r > 2$ it results:

$$\sum_{n=1}^{k} e_n(S_1 \star \ldots \star S_r) \le r \sum_{n=1}^{k} e_n(S_1) \star \ldots \star e_n(S_r),\ k = 1, 2, \ldots$$

This results from the fact that $e_{(n-1)r+1}(S_1 \star \ldots \star S_r) \le e_n(S_1) \star \ldots \star e_n(S_r)$ as follows:

$$\sum_{n=1}^{k} e_n(S_1 \star \ldots \star S_r) \le \sum_{n=1}^{rk} e_n(S_1 \star \ldots \star S_r) = \sum_{n=1}^{k} \sum_{i=(n-1)r+1}^{rn} e_i(S_1 \star \ldots \star S_r)$$

$$\le r \sum_{n=1}^{k} e_{(n-1)r+1}(S_1 \star \ldots \star S_r) \le r \sum_{n=1}^{k} e_n(S_1) \star \ldots \star e_n(S_r).$$

We can also prove the inequality

$$\prod_{n=1}^{k} e_n\left(\prod_{i=1}^{r} S_i\right) \leq \prod_{n=1}^{k}\prod_{i=1}^{r} e_n^r\left(S_i\right), \quad k=1,2,\ldots;\ r\geq 2.$$

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